Unifying the Dark QCD

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Declaration

I declare that the thesis entitled "**Unifying the Dark QCD**", submitted by me for the degree of Bachelor of Technology is the record of research work carried under the guidance of Prof. Daniel Stolarski, Carleton University, Canada and under the supervision of Prof. Binata Panda, Assistant Professor, Department of Physics, IIT(ISM) Dhanbad and had not formed the basis for award of any degree or diploma in any other university.

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Certificate

This is to certify that the work being presented in the Bachelors project report entitled "Unifying the Dark QCD" submitted in the Department of Physics, Indian Institute of Technology (Indian School of Mines), Dhanbad, India, is an authentic record of work carried out by **Mr. Chaitanya Pradeep Paranjape** (a final year student of B.Tech Engineering Physics), Admission No. 18JE0240, under the guidance of **Prof. Binata Panda**, Assistant Professor, Department of Physics, IIT (ISM) Dhanbad, and Prof. Daniel Stolarski, Associate Professor, Theoretical Particle Physics Group, Carleton University, Canada. The matter presented in this report has not formed the basis for the award of any degree or diploma in any other university.

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Abbreviations

SM	Standard Model
QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
GUT	Grand Unified Theory
SU(N)	Special Unitary group of rank N-1
EW	ElectroWeak
EWSB	ElectroWeak Symmetry Breaking
SSB	Spontaneous Symmetry Breaking
ir-rep	Ir-reducible representation of a group

Unifying the Dark QCD

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Abstract

This is the undergraduate thesis project report on 'Unifying the Dark QCD'. We assign the symmetry group of SU(3) for the 'Dark force interactions' and try to unify it with the Standard Model gauge group $SU(3) \otimes SU(2) \otimes U(1)$. We review the well established literature on Standard Model, Grand Unified Theories and Group Theoretical Techniques. We present the SU(8) group as a plausible starting point for the candidate unification group. We present an original study of the symmetry breaking of SU(8) to $SU(3) \otimes SU(3) \otimes SU(2) \otimes U(1)$ with the inclusion of cubic interaction for Higgs field in adjoint representation. We present some empirical evidence to support that the symmetry breaking of SU(8) to $SU(5) \otimes SU(3) \otimes U(1)$ or to $SU(3) \otimes SU(3) \otimes SU(2) \otimes U(1)$ is plausible in the given context.

Contents

1 Introduction

The recent discovery of Higgs boson [1],[2] corroborates the strong theoretical foundations of the Standard Model. The Higgs had been predicted since long accounting for the spontaneous symmetry breaking of the electroweak sector $SU(2) \otimes U(1)$. This remarkable achievement lays down the Standard model as our current best theory to explain the elementary particle interactions at high energies. However, there are still many open problems which motivate us to work with extensions of Standard model in search of beyond the Standard model physics. Accounting for neutrino oscillations and dark matter are just a few of them.

The nature of dark matter is probably one of the most sought after open problems in modern physics. The existence of dark matter has been confirmed from the astronomical observations but we still haven't been able to experimentally detect any dark matter candidate so far. We know the cold dark matter energy density in the universe is $\Omega_c h^2 = 0.120 \pm 0.001$ [3] which is almost five times of the baryon energy density $\Omega_b h^2 = 0.0224 \pm 0.0001$ [3]. To explain this cold dark matter energy density, there have been many theoretical advancements introducing prominent dark matter candidates like WIMPs, Axion which are actively being searched for (Refer [6] for review on dark matter candidates). Motivated by the comparable energy densities $\Omega_c \sim 5 \cdot \Omega_b$, we will base our work on the model of "Dark QCD" introduced by Y. Bai and P. Schwaller in [7]. The energy density for the ordinary matter is mostly accounted by the theory of strong interaction(QCD), which gives rise to Baryon masses related with $\Lambda_{\rm QCD}$ confinement scale in QCD. This motivates us to believe a similar dynamics with strong couplings should exist in the dark matter sector, and in order to produce comparable energy densities, they could have Baryon masses $m_D \sim m_p$ and number densitites $n_D \sim n_B$ of the same order. The theoretical models to explore these possibilities have been treated in [7].

Therefore, to assume an asymptotic-free QCD-like dynamics in the dark sector, it has been assigned the symmetry group $SU(N_d)$ identical to the color symmetry $SU(N_c)$ with $N_d = N_c = 3$. Our target during this project would be to unify the symmetry group for this 'dark force' with the symmetry group associated with the Standard model of particle physics $SU(3) \otimes SU(2) \otimes U(1)$. Please refer [8] for a review on Standard model and unification theories.

The rest of this thesis report is structured as follows. In section 2, we cover the fundamentals of the Standard Model of Particle Physics, and in the section 3 we review the motivation for the Grand Unified Theories along with the study of the well-known Georgi-Glashow SU(5) Model. In section 4 we go through the important group theoretical techniques which are heavily involved in Grand Unified Theory Model building. In section 5 we present our study of the SU(8) group for the required unification purposes, and we present our analysis of symmetry breaking with the inclusion of cubic interaction term for the Higgs field in adjoint representation. We finally conclude our work in section 6 and discuss ideas towards lines of future work.

2 Standard Model of Particle Physics

2.1 Gauge groups and Symmetry

Standard Model of Particle Physics is a gauge theory [14, 15, 16, 17], which is based on the symmetry of particle interactions characterised by the consequent symmetry groups. These symmetry groups are regarded as the gauge groups, described by the group theory. The gauge symmetry group of the Standard Model is built on the three gauge groups corresponding to the following different interactions:

$$G_{SM} : \mathrm{SU}(3)_C \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y. \tag{1}$$

The $\mathrm{SU}(3)_C$ is the strong interaction group, $\mathrm{SU}(2)_L$ is the weak interaction group, and the $\mathrm{U}(1)_Y$ is the Hypercharge group. In the next sub-section, we will shortly explain the role of these gauge groups in order to form a locally gauge invariant Lagrangian of the gauge theory.

The particle content can be expressed with Quantum numbers which represent the transformation properties of each particle under that specific gauge group. The force mediators i.e. vector boson fields of the Standard Model are represented as

$$\begin{aligned} G_\mu^a &= (8, 1, 0) \\ W_\mu^a &= (1, 3, 0) \\ B_\mu &= (1, 1, 0). \end{aligned} \tag{2}$$

while the matter content can be represented as the 15 right-handed Weyl fermions as follows :

$$\begin{aligned} u^\dagger &= (3, 1, 2/3) \\ d^\dagger &= (3, 1, -1/3) \\ e^\dagger &= (1, 1, -1) \\ \overline{\psi}^\dagger &= (\overline{3}, 2, -1/6) \\ \overline{l}^\dagger &= (1, 2, 1/2). \end{aligned} \tag{3}$$

The above transformations are for right-handed fermions[1], where $\overline{\psi}_1^\dagger = \overline{d}^\dagger$, $\overline{\psi}_2^\dagger = \overline{u}^\dagger$ and $\overline{l}_1^\dagger = \overline{e}^\dagger$, $\overline{l}_2^\dagger = \overline{\nu}^\dagger$. Where, the first number corresponds to representation under SU(3), such that the **3** is fundamental representation (transforming like a triplet under SU(3)), $\overline{3}$ is it's complex conjugation and **1** is said to transform as a singlet. Similarly, the second number is the transformation under SU(2) and the third number is the Hypercharge of U(1) transformation. The origin of this numbers is related to transformation of corresponding creation operators $a_{xr}^\dagger$ under the generators of the gauge group. We say it transforms in the representation (D, d, s) if it satisfies :

$$\begin{aligned} [T_a, a_{xr}^\dagger] &= a_{yr}^\dagger [T_a^D]_{yx} \\ [R_a, a_{xr}^\dagger] &= a_{xt}^\dagger [R_a^d]_{tr} \\ [S, a_{xr}^\dagger] &= s a_{xr}^\dagger \end{aligned} \tag{4}$$

Where the T_a are the 8 generators of the SU(3), R_a are 3 generators of the SU(2) and S is the lone generator of U(1) gauge group. The Cartan generators for SU(3) are T_3, T_8, for SU(2) it is the R_3 (weak iso-spin generator) and for U(1) it is the S generator. Adding them up it tells us that the Standard Model gauge group has a rank 4, and therefore, any group that we can hypothesize to

[1] The † indicates the creation operator for corresponding particle.

contain SM as its sub-algebra, must at least have rank 4. This just being the initial requirement, additionally, the required gauge group must also be able to have the required transformation algebra when decomposed from its ir-reducible representations. E.g. for right handed particles of standard model, their transformation algebra under the action of SM gauge group can be stated in one line as follows :

$$(1,1,-1) \oplus (3,1,2/3) \oplus (\overline{3},2,-1/6) \oplus (1,2,1/2) \oplus (3,1,-1/3) \tag{5}$$

For left handed particles, just take the complex conjugation (use $2 = \overline{2}$):

$$(1,1,1) \oplus (\overline{3},1,-2/3) \oplus (3,2,1/6) \oplus (1,2,-1/2) \oplus (\overline{3},1,1/3) \tag{6}$$

Therefore, the larger unifying gauge group we hope to achieve, must have its ir-reducible representations decomposable to transform according to the sub-algebra given by equations 5,6. For a detailed review on Standard Model please refer to [23].

2.2 Gauge invariance

The gauge theory is formed on the principle of gauge invariance. The principle dictates that the Lagrangian must be locally gauge invariant under the local transformations of the gauge symmetry groups. This gives rise to the need of vector bosons to mediate the force interactions. To understand the mathematical essence of the gauge invariance, consider the Lagrangian for the free Dirac fermion.

$$\mathcal{L}_{Dirac} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{7}$$

Now, under the global U(1) transformations, we can observe that the $\mathcal{L}_{\text{Dirac}}$ is clearly invariant.

$$\psi \to \psi' = e^{i\alpha Q}\psi \tag{8}$$

However, if we now consider the local U(1) transformation, i.e. $\alpha = \alpha(x)$, we have an extra term in the Lagrangian.

$$\mathcal{L}'_{Dirac} = \mathcal{L}_{Dirac} - Q\overline{\psi}\gamma^\mu \psi(\partial_\mu \alpha) \tag{9}$$

To get rid of this extra term, we introduce a vector field $A_\mu(x)$ and use it to define the Co-variant derivative.

$$D_\mu = \partial_\mu + ieQA_\mu \tag{10}$$

We want the co-variant derivative to stay invariant under these local U(1) transformations. Thus it must have the transformation properies as follows :

$$D'_\mu \psi' \to e^{i\alpha Q} D_\mu \psi \tag{11}$$

Accordingly, imposing this constraint gives us the transformation rule for the gauge field $A_\mu(x)$:

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \alpha \tag{12}$$

Therefore, now with this co-variant derivative, we can construct a Lagrangian which is invariant under the local gauge transformations.

$$\mathcal{L}_{Dirac} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{13}$$

We can now add the kinetic term for the gauge boson field, and then we have the QED Lagrangian, describing the electromagnetic interactions.

$$\mathcal{L}_{QED} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{14}$$

Where, the $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor for the Electromagnetic field. Thus, the principle of local gauge invariance automatically provides the interaction of Dirac fermions with the gauge fields, thus giving rise to the theory of QED which is a very successful gauge theory with unprecedented precision tests.

To generalise for an SU(N) group, the principle of local gauge in-variance gives rise to the (N^2-1) gauge fields and a single gauge field for the U(1) gauge group.

Similarly, we can construct the locally gauge invariant Largangians for the QCD sector, mediating strong interactions and the electroweak sector mediating the weak interactions.

$$\mathcal{L}_{QCD} = -\frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu} + \sum_f \overline{q_f}(i\gamma^\mu D_\mu) q_f \tag{15}$$

$$\mathcal{L}_{EW} = -\frac{1}{4} W_i^{\mu\nu} W^i_{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} + \sum_j \overline{\psi_j}(i\gamma^\mu D_\mu)\psi_j \tag{16}$$

For a detailed review on local gauge in-variance in the Standard Model please refer to [23].

2.3 Electroweak Symmetry Breaking

Let us explicitly consider the Electroweak sector G_{EW} : $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$. Let's consider the transformation for the family of quarks :

$$\psi_1(x) = \begin{pmatrix} u \\ d \end{pmatrix}_L \quad , \quad \psi_2(x) = u_R \ , \quad \psi_3(x) = d_R \tag{17}$$

or for the leptons.

$$\psi_1(x) = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad , \quad \psi_2(x) = \nu_{eR} \ , \quad \psi_3(x) = e^-_R \tag{18}$$

We know the $\mathcal{L}_{EW}$ is invariant under the following gauge transformations.

$$\psi_1(x) \to \psi_1'(x) = e^{iy_1\beta} e^{i\frac{\sigma_i}{2}\alpha^i} \psi_1(x) \tag{19}$$

$$\psi_2(x) \to \psi_2'(x) = e^{iy_2\beta} \psi_2(x) \tag{20}$$

$$\psi_3(x) \to \psi_3'(x) = e^{iy_3\beta} \psi_3(x) \tag{21}$$

where $\alpha^i = \alpha^i(x)$ and $\beta = \beta(x)$. The action of corresponding co-variant derivatives are :

$$D_\mu \psi_1(x) = \left[\partial_\mu - ig\frac{\sigma_i}{2} W^i_\mu - ig' y_1 B_\mu(x)\right] \psi_1(x) \tag{22}$$

$$D_\mu \psi_2(x) = \left[\partial_\mu - ig' y_2 B_\mu(x)\right] \psi_2(x) \tag{23}$$

$$D_\mu \psi_3(x) = \left[\partial_\mu - ig' y_3 B_\mu(x)\right] \psi_3(x) \tag{24}$$

where the g is the $\mathrm{SU}(2)_L$ gauge coupling and the g' is the $\mathrm{U}(1)_Y$ gauge coupling.

When we expand the QCD and the Electroweak Lagrangians, $\mathcal{L}_{QCD}$ and $\mathcal{L}_{EW}$, they give rise to various terms each having its own significance. Some correspond to the cubic and quartic self-interactions, while some correspond to the charged current interactions, neutral current interactions, etc. However, one can observe that it doesn't contain explicit mass terms for the gauge bosons and for the fermions. Adding a term like $\sim m^2 A^\mu A_\mu$ would make the Lagrangian lose its local gauge in-variance, the importance of which we have already established. If we add a fermion mass term, it would mix the left and right handed fields, violating local gauge invariance, as the transformation properties vary for the left and right handed fields.

Therefore, we need some clever way to account for the masses of our fundamental particles. This is done through the mechanism of spontaneous symmetry breaking, also widely known as the Higgs mechanism [2] [11].

For breaking the Electroweak sector G_{EW} : $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$, we consider an $\mathrm{SU}(2)_L$ doublet of complex scalar fields.

$$\phi(x) = \begin{pmatrix} \phi^{(+)}(x) \\ \phi^{(0)}(x) \end{pmatrix} \tag{25}$$

and construct a scalar lagrangian $\mathcal{L}_S$ which is invariant under the local G_{EW} : $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$ transformations. The scalar Lagrangian $\mathcal{L}_S$ can be written as :

$$\mathcal{L}_S = (D_\mu)^\dagger D^\mu \phi - \mu^2 \phi^\dagger \phi - h(\phi^\dagger \phi)^2 \tag{26}$$

where we have chosen the most general form of potential so as to preserve the renormalizability of the theory :

$$V(\phi^\dagger \phi) = \mu^2(\phi^\dagger \phi) + h(\phi^\dagger \phi)^2 \tag{27}$$

And the action of the co-variant derivative is :

$$D^\mu \phi = \left(\partial^\mu - ig\frac{\sigma^i}{2} W_i^\mu - ig' y_\phi B^\mu \right) \phi \tag{28}$$

For the potential to have bounded non-trivial minimas, we require $\mu^2 < 0$ and $h > 0$. Because the field is scalar, it can acquire a non-vanishing vacuum expectation value, which can be characterised as :

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{29}$$

We have parametrised in terms of the constant value v, based on the fact that we can always apply a global $\mathrm{SU}(2)_L$ transformation to rotate the doublet accordingly.

Now, in order to obtain global minima, we minimise the potential in terms of the variable v. We obtain :

$$v^2 = -\frac{\mu^2}{h} \tag{30}$$

One thing to note is that, when the Higgs doublet acquires the non-vanishing vacuum expectation value, it still transforms under the two generators of the $\mathrm{SU}(2)_L$. However, it transforms as a singlet under the combined action of weak iso-spin generator R_3 of the $\mathrm{SU}(2)_L$ and the Hypercharge generator S of the $\mathrm{U}(1)_Y$.

$$\implies (R_3 + S)\langle \phi \rangle = 0 \tag{31}$$

Thus, we identify it as the electromagnetic charge operator.

$$\therefore Q = R_3 + S \tag{32}$$

This lone diagonal generator is characterised with the $\mathrm{U}(1)_{em}$ gauge group of Electromagnetic interactions. Therefore, the $\mathrm{SU}(2)_L$ Higgs doublet is said to have broken the Electroweak sector

[2] Multiple authors contributed to the development of SSB mechanism. Namely the P. Higgs [11], F. Englert and R. Brout [12], G. Guralnik, C. Hagen, and T. Kibble [13].

$SU(2)_L \otimes U(1)_Y$ to the $U(1)_{em}$ gauge group of Electromagnetic interactions.

We can consider perturbations around this vacuum expectation value in the unitary (physical) gauge with the Higgs field.

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H \end{pmatrix} \tag{33}$$

Substituting this back into the expression for Lagrangian $\mathcal{L}_S$, we obtain the mass terms for the gauge boson fields. Specifically, we obtain the terms

$$\mathcal{L} \supset \frac{v^2}{8} \left(g^2 (W^1_\mu + W^2_\mu)^2 + (g W^3_\mu - g' B_\mu)^2 \right) \tag{34}$$

Digonalisation of the mass matrix of these vector bosons $(W^1_\mu, W^2_\mu, W^3_\mu, B_\mu)$ leads to definition of physical gauge bosons. Their mass terms are now explicitly given as :

$$M_W^2 W^+_\mu W^{-\mu} \ , \quad \frac{M_Z^2}{2} Z_\mu Z^\mu \tag{35}$$

where, the gauge boson fields are :

$$Z_\mu = B_\mu \cos\theta_W - W^3_\mu \sin\theta_W \tag{36}$$

$$A_\mu = B_\mu \cos\theta_W + W^3_\mu \sin\theta_W \tag{37}$$

$$W^\pm_\mu = \frac{W^1_\mu \pm i W^2_\mu}{\sqrt{2}} \tag{38}$$

Where θ_W is the electroweak mixing angle. The standard relations relating the Electroweak sector $SU(2)_L \otimes U(1)_Y$ and the $U(1)_{em}$ electromagnetic sector are :

$$M_Z \cos\theta_W = M_W = \frac{vg}{2}, \qquad m_H^2 = -2\mu^2 \tag{39}$$

$$g \sin\theta_W = g' \cos\theta_W = e, \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \qquad \cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}} \tag{40}$$

For brief reviews of the topics covered, please refer to [20, 21, 22, 24, 23].

3 Grand Unified Theories

3.1 Towards unification

Standard model of particle physics is an excellent description of elementary particles, in-fact the best description we have to offer, at least up to the electroweak energy scales. In fact, there are still many inadequacies of the Standard Model, and we need to think of ways to find SM extensions in order to include explanation for other physical phenomena in nature. Grand Unified Theories are a attractive place to start, ever since the success of electroweak sector proposed by Glashow-Salam-Weingberg [14, 15, 16, 17]. GUTs offer a unified description of the seemingly different physical processes, and reveal the hidden symmetry underlying the observed phenomena. Naturally, theorists are excited to explore the mathematically consistent GUTs that may answer at-least some of the open problems in fundamental physics. Let's shortly review some of the questions Standard Model is inadequate to answer :

- The Standard Model requires 18 free parameters to describe the intended physical processes. Why are there 18 free[3] parameters? Why not less? What is the minimum number of free parameters one can have for mathematically describing the physical description of reality?
- Why are there three generations of matter? Is there underlying symmetry between quarks and leptons?
- Why is the electroweak scale so small compared to the GUT scales: $M_{EW} \approx 10^{-17} M_{planck}$?
- Why shall there be three independent gauge symmetry groups? Are these three couplings related to each other in some fashion? Do they unify at a larger scale?
- Why is the Higgs mass so small? The radiative corrections to Higgs mass grow with the square of the cut-off scale, and in order to explain the the small scale of mass, we rely on 'fine-tuning'. The parameters are finely tuned to give the observed results, making it a naturalness problem.
- Standard Model does not provide information about the neutrino masses. Why are they extremely small yet finite? Is neutrino a majorana particle[4]?
- What is the Quantum description of theory of gravitational interactions? Does there exist a finite QFT for theory of quantized gravity? Why is Gravity so weak?
- What about the Dark matter and Dark energy? Why are the vacuum energy density predictions of Standard Model orders apart from the observed cosmological constant, with a opposite sign? What about the description of interaction and nature of Dark matter?

These are some of the most tempting questions that the physicists around the globe are trying to answer, the deepest underlying desire being to provide a complete mathematical description of the physical reality.

Therefore, we choose to go forward and try to include dark matter within the Standard Model by providing a Unified theory of both interactions. With the motivation already explained in section 1, we will assume an asymptotic-free QCD-like dynamics in the dark sector, which has been assigned the symmetry group SU(N_d) identical to the color symmetry SU(N_c) with $N_d = N_c = 3$. We look

[3] 6 quark masses, 3 lepton masses, 4 parameters for quark mixing matrix (CKM),vacuum expectation value of Higgs, Higss potential coupling parameter or Higgs mass, fine structure constant, electroweak mixing angle,strong coupling constant and 19th parameter for strong CP violation if one wants to include that.

[4] Majorana fermion is a fermion which is its own anti-particle.

forward to propose a Unified theory with a gauge group G such that it unifies the supposed SU(3) group of dark matter and the Standard Mode gauge group $\mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$. Mathematically, we are interested to identify $\mathrm{SU}(3) \otimes \mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$ as a sub-algebra of some larger group G :

$$G \supset \mathrm{SU}(3) \otimes \mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1) \tag{41}$$

This is the overall concept of Grand unified theories. For a detailed review, please refer to [8]. Since we don't observe these perfect symmetries directly, they are sometimes hidden through the mechanism of spontaneous symmetry breaking. Before we move on to try to propose our Unified theory, let's first review the famous Unified theory proposed by Georgi-Glashow, the SU(5) model [10]. It is a great starting point to explore the important aspects of GUTs.

3.2 The Georgi-Glashow SU(5) model

The SU(5) model of Grand Unified Theory for Standard Model was first proposed by Georgi-Glashow [10]. We need a group G that contains SM gauge group as sub-algebra.

$$G \supset \mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1) \tag{42}$$

The smallest group that satisfies this requirement is the SU(5) group, thus a minimal extension of the SM towards a GUT is based on the SU(5) group. In the Standard Model section 2.1, we have already stated the transformation properties of SM particle content under the SM symmetry groups. Equations 5,6 describe them particularly for the right handed set of particles and left handed set of particles. The goal is to represent the same transformation prpoerties under the decomposition of the ir-reps of the higher group chosen as a candidate for the unification. Let's see what are the decompositions of some of the ir-reps of SU(5). The $5, \overline{5}, 10, \overline{10}$ of SU(5) undergo following decomposition under the SM gauge group $\mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$.

$$\begin{aligned} 5 &\to (1, 2, 1/2) \oplus (3, 1, -1/3) \\ \overline{5} &\to (1, 2, -1/2) \oplus (\overline{3}, 1, 1/3) \\ 10 &\to (1, 1, 1) \oplus (\overline{3}, 1, -2/3) \oplus (3, 2, 1/6) \\ \overline{10} &\to (1, 1, -1) \oplus (3, 1, 2/3) \oplus (\overline{3}, 2, -1/6) \end{aligned} \tag{43}$$

We can observe that the direct sum of the $\overline{5}$ and 10 correspond identically with the transformation properties of the left handed fermions of Standard Model, refer in equation 6. Similarly, the direct sum of the 5 and $\overline{10}$ correspond identically with the transformation properties of the right handed fermions of Standard Model, refer in equation 5. Thus, we can conclude that the SM particle content can be arranged in a 5-plet[5] and 10-plet of the SU(5). We can do so in following fashion [8].

$$\overline{5} = \begin{pmatrix} d_g^C \\ d_r^C \\ d_b^C \\ e^- \\ -\nu_e \end{pmatrix} \qquad 10 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & +u_b^C & -u_r^C & -u_g & -d_g \\ -u_b^C & 0 & +u_g^C & -u_r & -d_r \\ +u_r^C & -u_g^C & 0 & -u_b & -d_b \\ +u_g & +u_r & +u_b & 0 & -e^+ \\ +d_g & +d_r & +d_b & +e^+ & 0 \end{pmatrix} \tag{44}$$

The local gauge invariance (refer to the section 2.2) requires that the interaction of matter fields are caused by $5^2 - 1 = 24$ gauge fields of the SU(5). The gauge fields transform in the adjoint representation of the SU(5), which can be written as 5×5 matrix

$$24 = \begin{pmatrix} G_{11} - \frac{2B}{\sqrt{30}} & G_{12} & G_{13} & X_1^C & Y_1^C \\ G_{21} & G_{22} - \frac{2B}{\sqrt{30}} & G_{23} & X_2^C & Y_2^C \\ G_{31} & G_{32} & G_{33} - \frac{2B}{\sqrt{30}} & X_3^C & Y_3^C \\ X_1 & X_2 & X_3 & \frac{W^3}{\sqrt{2}} + \frac{2B}{\sqrt{30}} & W^+ \\ Y_1 & Y_2 & Y_3 & W^- & -\frac{W^3}{\sqrt{2}} + \frac{2B}{\sqrt{30}} \end{pmatrix} \tag{45}$$

G, W, B are our regular SM gauge boson fields. We can see that there are new X and Y gauge boson fields, which can mix the quarks and leptons, and are thus famously called as the vector leptoquarks. We can see that X bosons couple to the electron and d-quark, and thus must cacrry 4/3 of electric charge. Similarly, the Y bosons must carry 1/3 of the electric charge.

The SU(5) group is certainly broken, as otherwise the leptoquarks would mediate proton decay reactions, but we know protons have a very high lifetime and thus the SU(5) gauge bosons X, Y

[5] The d_g^C refers to the complex conjugated particle.

have to be very heavy, $\sim 10^{15}$ GeV. As we will explain in section 4.3, the SU(5) can be broken to SU(3) ⊗ SU(2) ⊗ U(1) with the Higgs field in adjoint representation 24. It can be chosen as :

$$\langle \Phi_{24} \rangle = v \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \tag{46}$$

Due to the symmetry breaking, the X, Y vector bosons acquire mass by eating the 12 goldstone bosons. The mass of the new X, Y gauge bosons can be calculated as $M_X^2 = M_Y^2 = \frac{25}{2} g_5^2 v^2$. The SU(3) ⊗ SU(2) ⊗ U(1) is still unbroken and Φ_{24} is invariant under rotations of it. Thus, the W and Z bosons are still massless and we need to embed the Standard Model Higgs in the representation of SU(5) as well. As we can review from section 2.3, the Standard Model Higgs appears as a SU(2) doublet. Thus, the minimal way to embed it, is to use the fundamental representation of SU(5). We know that the 5 of SU(5) decomposes under SU(3) ⊗ SU(2) ⊗ U(1) as follows :

$$5 \to (1,2)(-3) \oplus (3,1)(2) \tag{47}$$

Thus, if we need to embed a SU(2) doublet, we need to introduce a SU(3) color triplet in the 5-plet for Higgs.

$$\langle \Phi_5 \rangle = \begin{pmatrix} T \\ H \end{pmatrix} \tag{48}$$

Where H signifies the SM Higgs doublet and the T is the colored Higgs triplet that we introduced. This triplet can induce reactions between quarks and leptons and allow for proton decay. But again, because we know protons have a very high lifetime, it forces us to have a very high mass for the colored Higgs triplet. A rough estimate for the mass of triplet requires it to be of order $\sim 10^{16}$ GeV, whereas we know that the weak doublet has a mass $\sim 10^2$ GeV. It is difficult to naturally explain the large mass gap between the triplet and doublet orginating from the same 5-plet. Thus, this gives rise to the fine tuning of the infamous doublet-triplet problem.

For brief reviews of the topics covered, please refer to [8, 21, 22].

Anyhow, this is the overall structure of the minimal-SU(5) GUT theory and its Higgs mechanism. One should write the Yukawa parameters and perform the calculations for quantifying any exact results. This theory serves as a good starting point for a basic study of the GUTs.

3.3 Unifying Standard Model with the Dark QCD

Now, we know that we need to find a group G which contains the SU(3) group of dark matter and the Standard Mode gauge group SU(3) ⊗ SU(3) ⊗ SU(2) ⊗ U(1).

$$G \supset \mathrm{SU}(3) \otimes \mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1) \tag{49}$$

The rank of SU(3) is 2, and SU(2) and U(1) are rank 1 groups. Therefore, the supposed group G must at-least have a rank of $2 + 2 + 1 + 1 = 6$. Because of the Classification Theorem [9], we know that we can choose the groups from either the 4 infinite families : A_n, B_n, C_n, D_n or the 5 exceptional algebras E_6, E_7, E_8, F_4 or G_2. In order to choose a rank 6 group, we can start from the groups SU(7), SO(12), SO(13), Sp(12) or opt for the exceptional algebras E_6, E_7, E_8. However, for now, we will try to work with a SU(N) group to find the unified gauge group.

We can choose SU(7) to be our candidate group, however, if we want to anticipate that we will need to perform the symmetry breaking in at-least three steps, we will need to account for an additional U(1) group[6] that will be generated. Thus, a rank 7 group will be required. Therefore, for the purposes of this study, we will choose to go forward with the SU(8) group being the candidate group and doing the complete analysis for that.

Alternatively, we can also start with SU(7) and break it directly to SU(3)⊗SU(3)⊗SU(2)⊗U(1), and then perform the Standard Model Higgs breaking mechanism. We will try to explore both of these avenues.

Before working with the SU(8) group directly, let's first familiarise ourselves with the various group theoretical techniques involved in Grand Unified Theory Model building.

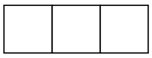
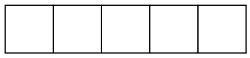
[6] Or an additional U(1) group for the dark charges of dark matter.

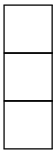
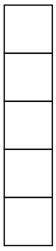
4 Group Theoretical Tools

4.1 Young Tableaux

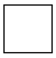
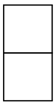
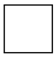
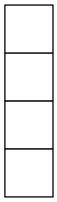
Young tableaux correspond to the ir-reducible representations (or tensors) of our symmetry groups e.g. SU(N). A legal Young tableau is defined as a collection of boxes such that from top to bottom, the row lengths are non-increasing, and from left to right, the column heights are non-increasing. Young tableaux are useful way to work with the ir-reps of desired groups. Let's see a few examples covering the important aspects of Young tableaux.

A rank-n totally Anti-symmetric tensor is represented as a single column of n boxes. While a rank-n totally symmetric tensor is represented as a single row of n boxes. Some examples :

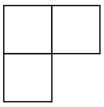
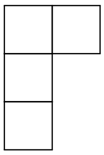
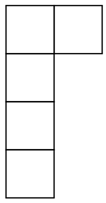
Rank-3 symmetric tensor :  Rank-5 symmetric tensor : 

Rank-3 Anti-symmetric tensor :  Rank-5 Anti-symmetric tensor : 

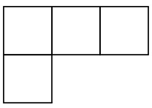
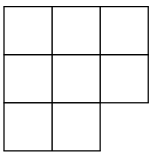
Let's concern ourselves with the ir-reps of SU(N). The fundamental representation N is just given as a single box. While the $\overline{N}$ representation is given as a single column of $N-1$ boxes.

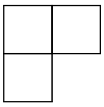
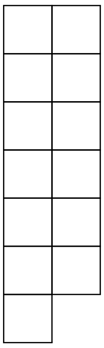
3 of SU(3) :  $\overline{3}$ of SU(3) :  5 of SU(5) :  $\overline{5}$ of SU(5) : 

While the Young tableaux of the adjoint ir-rep is given as a single column of $N-1$ boxes, but now with 2 boxes in the first row. The examples for adjoint ir-reps of some Young tableaux are :

3 of SU(2) :  8 of SU(3) :  15 of SU(4) :  24 of SU(5) : 

Let's see how we can find the Young tableaux of the complex conjugated ir-rep, from the young tableau of a given ir-rep. First fill up the necessary boxes required to complete the young tableau and form a rectangle of height N and keeping the width same. Now, remove the young tableau diagram of the given ir-rep from this rectangle. Rotate the remaining figure clockwise by 180^{deg}. The resulting tableau is the Young tableau of the complex conjugated ir-rep. Let's look at some examples :

45 of SU(4) :  $\overline{45}$ of SU(4) : 

168 of SU(8) :  $\overline{168}$ of SU(8) : 

The dimension of the ir-rep can be found from its Young tableau. Suppose we have a Young tableau of an ir-rep of SU(N). To calculate the dimension, we will assign two factors to each of the box. First, let's understand the numerator factor for each box. The numerator factor is given as $N + d_{box}$. Where the d_{box} increases from 0 as you go to the right, and decreases from 0 as you go down. The following diagrams represents the values of d_{box} as they shall appear in a young tableau.

0	1	2	3
-1	0	1	2
-2	-1	0	1

The denominator factor is called as the hook length of the box. Hook length of the box is the number of boxes that are below the given box plus the number of boxes to the right of the given box plus one, including the given box as well. The hook length factors for all the boxes have been written down form a few Young tableaux :

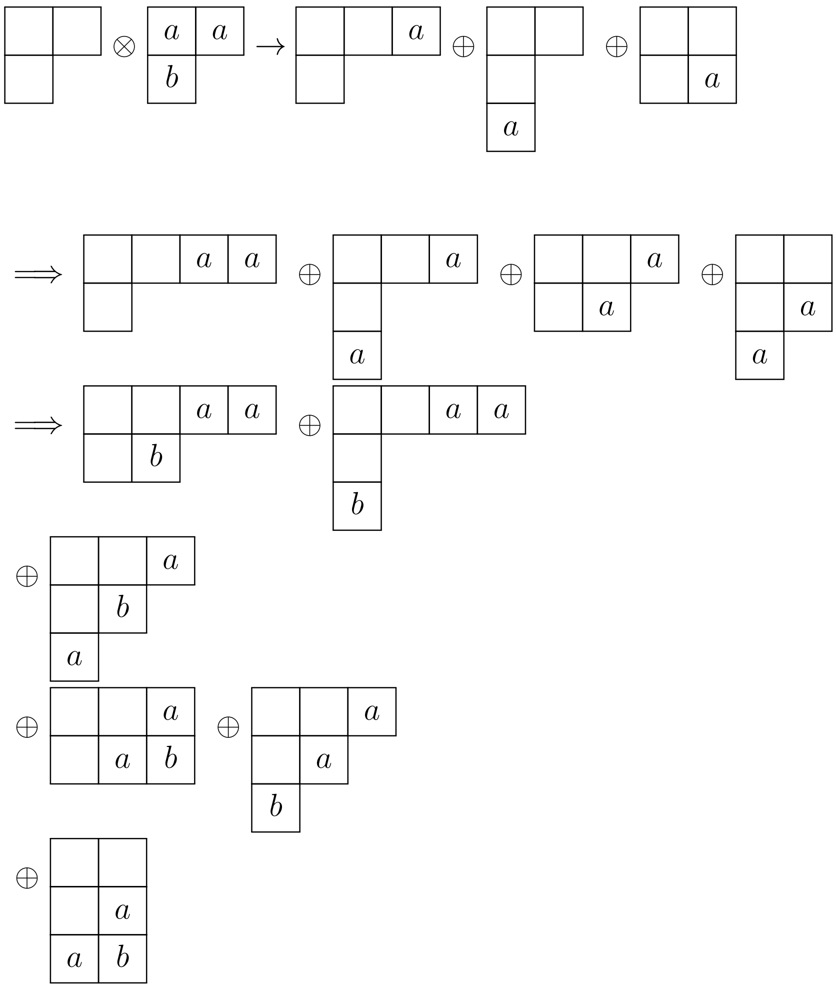
6	4	2	1
3	1		
1			

5	1
3	1
2	
1	

Now, we can calculate the $(\frac{N+d_{box}}{\text{hook length}})$ factors for all of the boxes. Then the dimension of the ir-rep represented by given Young tableau, is just the product of these factors for all of the boxes. Mathematically speaking :

$$\dim(\text{ir-rep}) = \prod_{all\ boxes} \frac{N + d_{box}}{\text{hook length}} \tag{50}$$

We can use the Young tableaux for decomposing the tensor products of ir-reps. To do so, write the tensor product in young tableaux, and for the second young tableau, fill up the first row with as, fill up the second row with bs, and so on. Now, take the single box with a from the second young tableau and join it to the first tableau in all possible legal ways. Repeat this with all the a-boxes of the second tableau, joining them to the updated tableaux. Note that two a-boxes must not be in the same column. Perform the same procedure with b-boxes and so on. In the end, collect all the legal tableaux, and read the entries of as, bs, ... from right to left and from top row to bottom. While reading the entries, the number of as must be greater than or equal to the number of bs, at all times. Otherwise, discard that tableau. Thus, finally we have the collection of all the Young tableaux which will be present in the decomposition of the product of given ir-reps. For a complete illustration, review the example of $8 \otimes 8$ decomposition of the SU(3) ir-reps.



We can identify the resulting tableaux with the ir-reps of SU(3). Thus we have proved : $8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10}$

Similarly, we can also calculate decomposition of an ir-rep to the lower rank gauge groups whose sub-algebra is contained in the original group. For this project, we will calculate such decompositions using the LieART Mathematica package[26, 27]. For example, decomposing the $\overline{10}$ of SU(5) to SU(3) $\otimes$ SU(2 $\otimes$ U(1 is given by the following command :

```
DecomposeIrrep[Irrep[SU5][Bar[10]], ProductAlgebra[SU3, SU2, U1]]
```

We can also calculate the decompositions of products of Ir-reps using the same LieART Mathematica package. The command to calculate the above $8 \otimes 8$ product is :

```
DecomposeProduct[Irrep[SU3][8], Irrep[SU3][8]]
```

We will be working with these commands to produce most of the calculations required during the project.

For a detailed stud of Young Tableaux, please refer to the [9].

4.2 Dynkin Diagrams

As explained earlier, the symmetries corresponding to the elementary particle interactions are best represented in the language of group theory, particularly with the Lie groups. Thus, understanding these Lie groups structure is of paramount importance in group theoretical study. The structure of the Lie groups are best written in a Cartan-Weyl basis, which is a maximal set of commuting generators called the Cartan Generators. The number of Cartan generators i.e. the maximum number of simultaneously diagonalizable generators of the Lie group, is the rank of that Lie group. Thus, for a Lie group with rank l, we have the Cartan generators basis $= \{H_1, \cdots, H_l\}$.

$$[H_i, H_j] = 0 \tag{51}$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{52}$$

$$[E_\alpha, E_\alpha] = \alpha \cdot H \tag{53}$$

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \tag{54}$$

The dimension of SU(N) group is N^2-1, thus the total number of generators is N^2-1. However, the rank of the SU(N) group is $N-1$. Therefore, SU(3) has two Cartan generators, SU(4) has 3 Cartan generators, and so on. The E_α correspond to the remaining generators of the algebra. The structure constants that appear in the commutation relation with the Cartan Generators are called the roots of the algebra. For a group with rank l, each root has l components, and we say that the root lives in an l-dimensional Euclidean space. For the group $SU(N)$, we have total of N^2-1 roots, out of which $N-1$ roots correspond to the zero vectors. These are the roots corresponding to the Cartan Generators. Additionally, out of the remaining non-zero roots, we can choose $N-1$ positive roots[7] such that, they cannot be expressed as a linear combination of the other positive roots. These particular $(N-1)$ positive roots are called the simple roots of the algebra. For a group with rank l, we have l simple roots. It can be further proved that [28] these simple roots actually span the l-dimensional Euclidean space that all the roots live in - root space. Thus, if we know the structure of the simple roots of an algebra, we can express all the roots as a linear combination of them. Consequently, with can find all the roots and completely retrieve the Lie algebra with the 4 equations above. For example let's look at the root structure of SU(3). The roots of SU(3) in the Cartan-weyl basis (H_1, H_2) are :

$$(1,0),\ (\frac{1}{2}, \frac{\sqrt{3}}{2}),\ (-\frac{1}{2}, \frac{\sqrt{3}}{2}),\ (-1,0)\ (-\frac{1}{2}, -\frac{\sqrt{3}}{2}),\ (\frac{1}{2}, -\frac{\sqrt{3}}{2}) \tag{55}$$

These roots can be graphically represented in the (H_1, H_2) plane, where we can observe that choosing the α_1 and α_2 positive roots to be the simple roots, we can express all the other roots as combination of $\alpha_1.\alpha_2$. Please refer to the figure 1.

Simple roots have some more interesting properties, which make them ideal to study the Lie Algebra of the groups. These simple roots of Lie algebra can be conveniently expressed in pictorial diagram, formally known as the Dynkin Diagram. A simple root is represented as a hollow circle in a Dynkin diagram. The lines connecting these circles signify the angle between the two corresponding simple roots. It has been proved that the angle between the simple roots can only take values of $90^\circ, 120^\circ, 135^\circ$, and 150° [9, 18]. Additionally, simple roots of a Lie algebra can have at most 2 different lengths. The angle between the two simple roots α, β can be written as

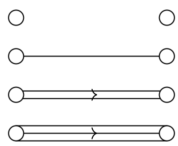
$$\cos^2\theta_{\alpha\beta} = \frac{(\alpha\cdot\beta)^2}{\alpha^2\beta^2} = \frac{(p-q)(p'-q')}{4} \tag{56}$$

[7] Positive roots are the roots with the first entry of root vector being positive.

where, the $((p-q)(p'-q'))$ must be a non-negative integer and hence the $\cos^2\theta_{\alpha\beta}$ can take on only 4 values, $\{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{3}\}$, corresponding to $90^{\circ}, 120^{\circ}(\text{or}60^{\circ}), 135^{\circ}(\text{or}45^{\circ})$ and $150^{\circ}(\text{or}30^{\circ})$. With further restrictions on lengths [18], we can can have at most 4 possibilities for a pair of simple roots as follows :

- Angle of 90°, with no restriction on length ratio
- Angle of 60° or 120°, with a length ratio of 1
- Angle of 45° or 135°, with a length ratio of $\sqrt{2}$
- Angle of 30° or 150°, with a length ratio of $\sqrt{3}$

The Dynkin diagrams for the above 4 possibilities of pair of roots are given as follows. (With the arrow pointing towards the root with shorter length[8]).



We are concerned with the Dynkin Diagrams of the SU(N) algebra which are given by the A_{N-1} algebra. An interested reader can refer to the other types of algebra[9] by referring to [9, 18]. The Dynkin Diagram of the A_n algebra is given as the n open circles connected with single line.

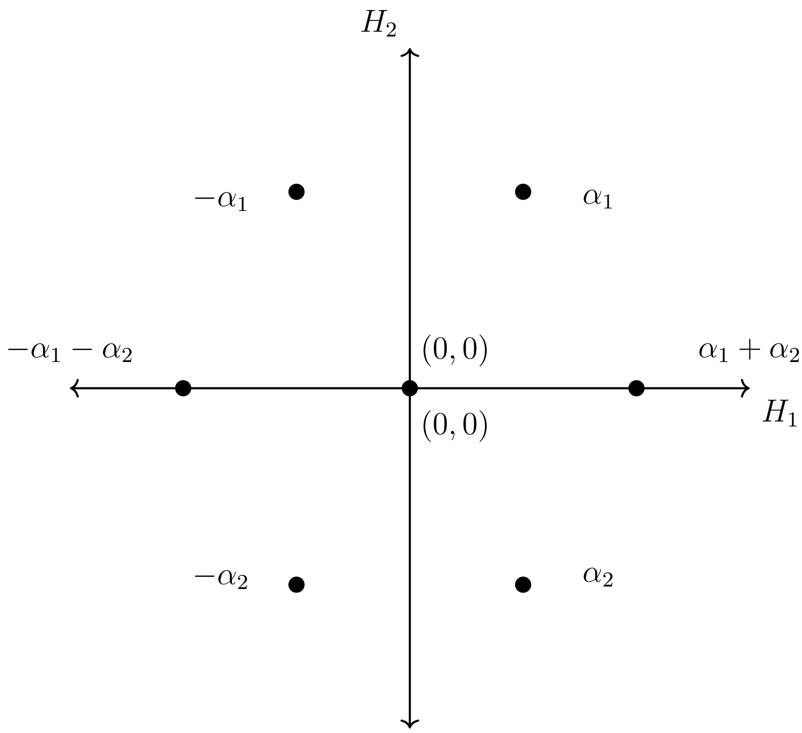
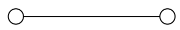


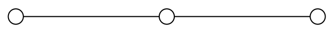
Figure 1: SU(3) roots structure

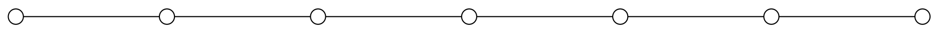
[8]Sometimes instead of the arrow, one draws a filled circle to represent the root with shorter length
[9]There are 4 infinite families of algebra, A_n, B_n, C_n, D_n and the 5 exceptional algebras E_6, E_7, E_8, F_4, G_2.

Therefore, the Dynkin Diagrams of SU(2), SU(3), SU(4) and SU(8) respectively, are given as follows :

SU(2) : 

SU(3) : 

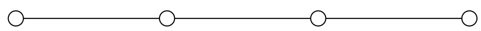
SU(4) : 

SU(8) : 

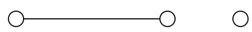
Dynkin diagrams are useful to identify the sub-algebras of given algebra, and thus trying to obtain the symmetry group of SM from the given group. Allowing for the possibilities of the various 'paths' of symmetry breaking to the SM. Sub-algebras can be of two types, regular and special. The regular sub-algebras can be obtained directly by looking at the Dynkin Diagrams. Interested reader is referred to the [18] for more about the special sub-algebras.

Regular sub-algebras are a subset of the given Lie-algebra. Thus, the simple roots of regular sub-algebra are also the subset of the simple roots of the given Lie-algebra. Therefore, leaving out one or multiple hollow circles from the Dynkin diagram of the given Lie-algebra gives us the Dynkin diagram of the Regular sub-algebra. Note that with this method, we can only obtain the sub-algebras with a rank lower than the rank of given Lie-algebra.

There are also maximal regular sub-algebra whose rank is equal to the rank of the given Lie-Algebra. To obtain such sub-algebras, we first construct an extended Dynkin diagram of the given Lie-algebra, by adding an additional hollow circle in the diagram. Every Dynkin Diagram has a unique extended Dynkin diagram. To find these extended diagrams, please refer to the [9]. To show an example of the paths of symmetry breaking, consider the Dynkin Diagram of SU(5) :

SU(5) : 

Note that U(1) groups are generated, every time we remove a hollow circle from the Diagram. Now observe that if we remove the third circle, we obtain the Standard model gauge group.

$SU(3) \otimes SU(2) \otimes U(1)$: 

Obtaining such direct paths of symmetry breaking by removing the circles from Dynkin Diagrams, is valid only if we work in adjoint representation of Higgs field. For more details, please refer to the [18].

We end here our short review of the Dynkin diagrams and the Lie algebra, an interested reader is referred to the Georgi and Slansky [9, 18, 20].

4.3 Symmetry breaking of SU(N) in adjoint representation

The detailed analysis for the Spontaneous Symmetry Breaking of SU(N) group for various representation is done by L.F. Li in [19]. Here, we will focus on the SSB in the adjoint ir-rep. [19] covers the case without the cubic interaction, however, we will soon see that we will need the cubic interaction analysis. For a brief review, one can also refer to the [21].

Let's consider the breaking of SU(N) with Higgs field in adjoint representation, without the cubic interaction. The most general form for a renormalizable potential is :

$$V(\Phi) = \frac{-\mu^2}{2} \cdot \mathrm{Tr}(\Phi^2) + \frac{\lambda_1}{4} \cdot \mathrm{Tr}(\Phi^4) + \frac{\lambda_2}{4} \cdot (\mathrm{Tr}(\Phi^2))^2 \tag{57}$$

We have freedom to choose vacuum, because with group transformation, we can go from one vacuum to another, and the potential will not change. We choose the vacuum such that the vacuum expectation value of Higgs field is in the block diagonal form. And thus, the broken generators do not commute with the vacuum expectation value of Higgs field. Suppose that the vacuum expectation value is diagonal with v_1 occurring N_1 times, v_2 occurring N_2 times and so on, with $\sum_i N_i v_i = 0$. The generators whose all non-zero entries lie within the ith block, commute with the vacuum expectation value, and thus form an unbroken symmetry group SU(N_i). Additionally, linear combination of diagonal generators which is proportional to vacuum expectation value, commutes with it to form U(1) subgroup. Thus, we can have symmetry breaking from SU(N) to SU(N_1)$\otimes$ SU(N_2) $\otimes \cdots$ U(1).

So let's consider the diagonal form of field :

$$\Phi = v \cdot \mathrm{diag}(\alpha_1, \cdots, \alpha_N) \tag{58}$$

Imposing a trace-less condition and normalisation :

$$\sum \alpha_i = 0 \quad \sum \alpha_i^2 = 1 \tag{59}$$

Substituting into potential we get,

$$V(\Phi) = \frac{-\mu^2}{2} \cdot v^2 + \frac{\lambda_1}{4} \cdot v^4 \sum \alpha_i^4 + \frac{\lambda_2}{4} \cdot v^4 \tag{60}$$

We can define coefficients :

$$A = \lambda_1 \sum \alpha_i^4 + \lambda_2 \quad C = -\mu^2 \tag{61}$$

So that Potential is written as :

$$V(\Phi) = A \cdot \frac{v^4}{4} + C \cdot \frac{v^2}{2} \tag{62}$$

To obtain the minima condition, we set $\frac{\partial V}{\partial v} = 0$ and omitting the trivial $v = 0$ solution, we obtain Quadratic in v with solution given by :

$$v^2 = -\frac{C}{A} \tag{63}$$

Substituting back into Potential we get,

$$V = -\frac{C^2}{4A} \tag{64}$$

where $A > 0, C < 0$. (For a bounded potential with non-trivial minimas)

We can see that the global minima is achieved when A is minimum. Thus, we shall minimise the function A with the trace-less and normalisation conditions. Using Lagrange multipliers :

$$f(\alpha_1, \alpha_2, \cdots, \alpha_{N-2}, a, b) = \lambda_2 + \lambda_1 \sum \alpha_i^4 - b\Big(\sum \alpha_i\Big) - a\Big(\sum \alpha_i^2 - 1\Big) \tag{65}$$

Taking partial derivative for equation for extreme :

$$\frac{\partial f}{\partial \alpha_j} = 4\lambda_1 \alpha_j^3 - 2a\alpha_j - b = 0 \tag{66}$$

$$\sum \alpha_i = 0 \qquad \sum \alpha_i^2 = 1 \tag{67}$$

We observe that it is a cubic equation, signifying there can be at max three different values of α_j. One can prove that the global minima is achieved when there are two different values on the diagonal[].

Now, let us assume that the global minima is obtained in the case where there are only two distinct values of diagonal entries. Such that, the value v_1 appears N_1 times and v_2 appears N_2 times, where N_1, N_2 are integers. We impose the $\sum \alpha_i = 0 \quad \sum \alpha_i^2 = 1$ conditions by requiring following to be true :

$$N_1 v_1 + N_2 v_2 = 0 \quad \text{AND} \quad N_1 v_1^2 + N_2 v_2^2 = 1 \tag{68}$$

The solution for the above two equations is given as :

$$v_1 = \frac{-N_2}{\sqrt{N \cdot N_1 \cdot N_2}},\ v_2 = \frac{N_1}{\sqrt{N \cdot N_1 \cdot N_2}} \quad \text{OR} \quad v_1 = \frac{N_2}{\sqrt{N \cdot N_1 \cdot N_2}},\ v_2 = \frac{-N_1}{\sqrt{N \cdot N_1 \cdot N_2}} \tag{69}$$

Where $N = N_1 + N_2$. In either case, the value of $\sum \alpha_i^4$ can be simplified to be as follows :

$$\sum \alpha_i^4 = N_1 v_1^4 + N_2 v_2^4 = v_1^2 + v_1 \cdot v_2 + v_2^2 = \frac{1}{N} \cdot \big(\frac{N_1}{N_2} + \frac{N_2}{N_1} - 1\big) \tag{70}$$

We can see that the A is now given as

$$A = \lambda_2 + \frac{\lambda_1}{N} \cdot \big(\frac{N_1}{N_2} + \frac{N_2}{N_1} - 1\big) \tag{71}$$

. Taking λ to be constants, the minima is obtained if $N_1 = N_2$. For even N it can be achieved exactly ($N_1 = N_2 = N/2$) or in the odd N case, minima is obtained for $\frac{N\pm1}{2}$ values of N_1 and N_2.

So, in the adjoint irrep of SU(N) without the cubic term, would break it to SU(N_+) $\times$ SU(N_-) $\times$ U(1), where $N_+ = \frac{N+1}{2}, N_- = \frac{N-1}{2}$ for odd N and $N_+ = N_- = \frac{N}{2}$ for even N.

We can see that if we want to break SU(8) with same procedure, we will break it to SU(4)⊗SU(4)⊗U(1). Thus, if we would like to break SU(8) to say SU(5)⊗SU(3)⊗U(1), we would have to adapt some different procedure.

A summary of various symmetry breaking results can be found in [19].

4.4 Anomaly coefficients

While working with the ir-reps of SU(N) groups and assigning them to explain transformation properties of particles, we must ensure that we are forming an anomaly-free consistent theory. This is ensured by checking that the sum of the triangular anomaly coefficients[10] of ir-reps used in representing particle content, add up to zero.

If we know the decompositions of products of ir-reps, we can use it to calculate the anomaly coefficients of other ir-reps. The following formulas will be of help in this task :

$$\mathcal{A}_3(R) = -\mathcal{A}_3(\overline{R}) \qquad \text{and } \mathcal{A}_3(N) = 1 \text{ where N is fundamental ir-rep of SU(N)} \tag{72}$$

$$\mathcal{A}_3(R_1 \oplus R_2) = \mathcal{A}_3(R_1) + \mathcal{A}_3(R_2) \tag{73}$$

$$\mathcal{A}_3(R_1 \otimes R_2) = dim(R_1) \cdot \mathcal{A}_3(R_2) + dim(R_2) \cdot \mathcal{A}_3(R_1) \tag{74}$$

Note that the anomaly coefficients of the real representations are thus zero. We can also calculate these coefficients by decomposing the SU(N) ir-reps to the SU(N-1) ⊗ U(1) such that :

$$R \to (r_i, f_i) \oplus (r_2, f_2) \oplus \cdots \tag{75}$$

and then the the triangular anomaly coefficients are calculated as :

$$\mathcal{A}_3(R) = \sum_i I_2(r_i) \cdot f_i \tag{76}$$

where the $I_2(r_i)$ is the second-order index of the ir-rep r_i of SU$(N-1)$, related to the Quadratic Casimir Invariant. Refer to the [25].

Let's see the example of SU(5) ir-reps. Let's decompose the 10 of SU(5) to SU(4) ⊗U(1). We do so in the LieART with the command DecomposeIrrep[Irrep[SU5][10], ProductAlgebra[SU4, U1]], we obtain :

$$10 \to (4, -3) \oplus (6, 2) \tag{77}$$

The second-order index of the ir-reps 4 and 6 of SU(4) are obtained from LieART with the command Index[Irrep[SU4][6]]. The index of 4 is 1 as expected and of 6 is 2. Substituting we obtain the index of 10 of SU(5) to be equal to 1 as well. Therefore, we can see that the $5+\overline{10}$ or the $\overline{5}+10$ is anomaly free.

$$\mathcal{A}_3(5 \oplus \overline{10}) = \mathcal{A}_3(\overline{5} \oplus 10) = 0 \tag{78}$$

Thus, SU(5) is a anomaly free consistent theory. In this way, we can find the anomaly coefficients of ir-reps with group theory, and check the consistency of our physical theory.

For a brief review on the anomaly coefficients, please refer to the [9, 29].

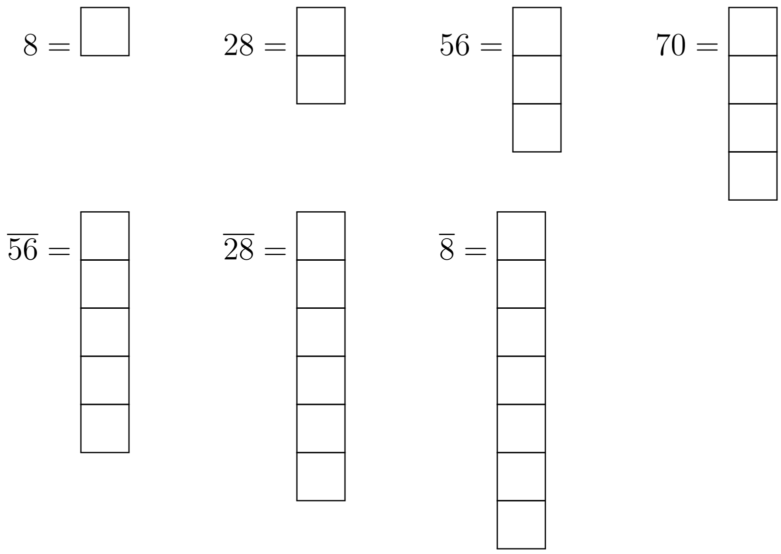
[10] In a 4-dimensional theory, we are concerned with the traingular anomaly coefficients. For more information, please refer to the [29, 25].

5 The SU(8) model

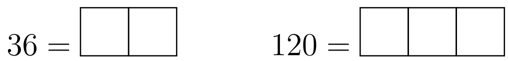
5.1 SU(8) ir-reps

The Anti-symmetric ir-reps of the SU(8) will have the dimensionalities given by the binomial coefficients : $\binom{8}{1}$, $\binom{8}{2}$, $\binom{8}{1}$, $\cdots$, $\binom{8}{7}$.

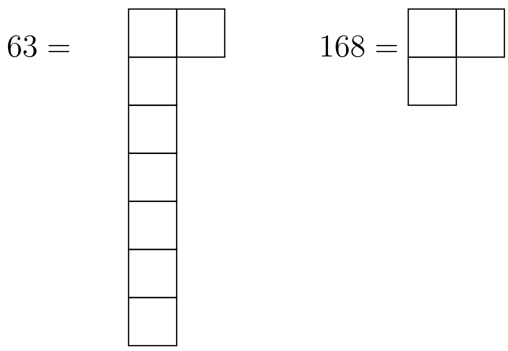
The corresponding Young Tableaux can be constructed by adding boxes in the single column upto (8-1) boxes. These can be written as :



The symmetric ir-reps can be constructed by adding boxes into a single row. The dimensionalities will be $\frac{8\cdot 9}{2}$, $\frac{8\cdot\cdot 10}{3\cdot 2}$, $\cdots$. The corresponding Young Tableaux are :



Young Tableaux of some other useful ir-reps :



We have considered ir-reps with dimensionalities only up-to 200.

Let us also calculate the group theory triangular anomaly coefficients of these ir-reps. We decompose the SU(8) ir-reps to the SU(7) $\otimes$ U(1) such that :

$$R \to (r_i, f_i) \oplus (r_2, f_2) \oplus \cdots \tag{79}$$

and then the the triangular anomaly coefficients are calculated as :

$$\mathcal{A}_3(R) = \sum_i I_2(r_i) \cdot f_i \tag{80}$$

where the $I_2(r_i)$ is the second-order index of the ir-rep, related to the Quadratic Casimir Invariant. The $I_2(r_i)$ is obtained from LieART with the command Index[Irrep]. The decompositions of the above ir-reps to SU(7) ⊗ U(1) can be calculated in LieART with command DecomposeIrrep[Irrep[SU8][28], ProductAlgebra[SU7, U1]]. Replace 28 with other ir-reps to obtain the following list :

$$\begin{aligned}
8 &\to (1,-7) \oplus (7,1) \\
28 &\to (7,-6) \oplus (21,2) \\
36 &\to (1,-14) \oplus (7,-6) \oplus (28,2) \\
56 &\to (21,-5) \oplus (35,3) \\
63 &\to (1,0) \oplus (48,0) \oplus (7,8) \oplus (\overline{7},-8) \\
70 &\to (35,-4) \oplus (\overline{35},4) \\
120 &\to (1,-21) \oplus (7,-13) \oplus (28,-5) \oplus (84,3) \\
168 &\to (7,-13) \oplus (21,-5) \oplus (28,-5) \oplus (112,3)
\end{aligned} \tag{81}$$

We can go ahead and calculate the coefficients as per the described procedure. We obtain :

$$\begin{aligned}
\mathcal{A}_3(8) &= 1 \\
\mathcal{A}_3(28) &= 4 \\
\mathcal{A}_3(36) &= 12 \\
\mathcal{A}_3(56) &= 5 \\
\mathcal{A}_3(63) &= 0 \\
\mathcal{A}_3(70) &= 0 \\
\mathcal{A}_3(120) &= 77 \\
\mathcal{A}_3(168) &= 55
\end{aligned} \tag{82}$$

While we know that $\mathcal{A}_3(R) = -\mathcal{A}_3(\overline{R})$.

Now that we have good information about the ir-reps of $SU(8)$, let's decompose it in various ways to see if we can obtain the desired gauge group embeddings.

5.2 SU(8) decompositions

Let's write down the decompositions of SU(8) irreps to $\mathrm{SU}(5) \otimes \mathrm{SU}(3) \otimes \mathrm{U}'(1)$

$$
\begin{aligned}
(1) \quad & 8 \to (1,3)(-5) \oplus (5,1)(3) \\
(4) \quad & 28 \to (1,\overline{3})(-10) \oplus (5,3)(-2) \oplus (10,1)(6) \\
(12) \quad & 36 \to (1,6)(-10) \oplus (5,3)(-2) \oplus (15,1)(6) \\
(5) \quad & 56 \to (1,1)(-15) \oplus (5,\overline{3})(-7) \oplus (\overline{10},1)(9) \oplus (10,3)(1) \\
(0) \quad & 63 \to (1,1)(0) \oplus (5,\overline{3})(8) \oplus (\overline{5},3)(-8) \oplus (1,8)(0) \oplus (24,1)(0) \\
(0) \quad & 70 \to (5,1)(-12) \oplus (\overline{5},1)(12) \oplus (10,\overline{3})(-4) \oplus (\overline{10},3)(4) \\
(77) \quad & 120 \to (5,6)(-7) \oplus (1,10)(-15) \oplus (15,3)(1) \oplus (\overline{35},1)(9) \\
(55) \quad & 168 \to (5,\overline{3})(-7) \oplus (1,8)(-15) \oplus (5,6)(-7) \oplus (10,3)(1) \oplus (15,3)(1) \oplus (\overline{40},1)(9) \\
(-55) \quad & \overline{168} \to (\overline{5},3)(7) \oplus (1,8)(15) \oplus (\overline{5},\overline{6})(7) \oplus (\overline{10},\overline{3})(-1) \oplus (\overline{15},\overline{3})(-1) \oplus (40,1)(-9) \\
(-77) \quad & \overline{120} \to (\overline{5},\overline{6})(7) \oplus (1,\overline{10})(15) \oplus (\overline{15},\overline{3})(-1) \oplus (35,1)(-9) \\
(-5) \quad & \overline{56} \to (1,1)(15) \oplus (\overline{5},3)(7) \oplus (10,1)(-9) \oplus (\overline{10},\overline{3})(-1) \\
(-12) \quad & \overline{36} \to (1,\overline{6})(10) \oplus (\overline{5},\overline{3})(2) \oplus (\overline{15},1)(-6) \\
(-4) \quad & \overline{28} \to (1,3)(10) \oplus (\overline{5},\overline{3})(2) \oplus (\overline{10},1)(-6) \\
(-1) \quad & \overline{8} \to (1,\overline{3})(5) \oplus (\overline{5},1)(-3)
\end{aligned}
\tag{83}
$$

The number (#) appearing before each representation is the group theory triangular anomaly coefficient for that particular representation. In order to have an anomaly free theory, we must combine ir-reps whose sum of anomaly coefficients is zero. The first plausible combination satisfying this condition is $8 \oplus 28 \oplus \overline{56}$. However, if we observe their decomposition to $\mathrm{SU}(5) \otimes \mathrm{SU}(3) \otimes \mathrm{U}'(1)$:

$$
\begin{aligned}
(1) \quad & 8 \to (1,3)(-5) \oplus (5,1)(3) \\
(4) \quad & 28 \to (1,\overline{3})(-10) \oplus (5,3)(-2) \oplus (10,1)(6) \\
(-5) \quad & \overline{56} \to (1,1)(15) \oplus (\overline{5},3)(7) \oplus (10,1)(-9) \oplus (\overline{10},\overline{3})(-1)
\end{aligned}
\tag{84}
$$

We can observe that we don't have a combination of $5 \oplus \overline{10}$ in the SU(5) sector with a simultaneous SU(3) dark singlet, which we have already established that corresponds to SM transformation. Another plausible combination should be $3 \cdot (28) \oplus \overline{36}$, but it suffers from the same problem. Probably the only combination with the SM particle contents should be the $2 \cdot (8 \oplus 56) \oplus \overline{36}$. Let's write down the corresponding decomposition :

$$
\begin{aligned}
(1) \quad & 8 \to (1,3)(-5) \oplus (5,1)(3) \\
(5) \quad & 56 \to (1,1)(-15) \oplus (5,\overline{3})(-7) \oplus (\overline{10},1)(9) \oplus (10,3)(1) \\
(-12) \quad & \overline{36} \to (1,\overline{6})(10) \oplus (\overline{5},\overline{3})(2) \oplus (\overline{15},1)(-6)
\end{aligned}
\tag{85}
$$

Let's also decompose the SU(5) sector to Standard Model gauge group.

$$
\begin{aligned}
(1,3)(-5) &\to (1,1,3)(0)(-5)\\
*\,(5,1)(3) &\to (1,2,1)(-3)(3) \oplus (3,1,1)(2)(3)\\
(1,1)(-15) &\to (1,1,1)(0)(-15)\\
(5,\overline{3})(-7) &\to (1,2,\overline{3})(-3)(-7) \oplus (3,1,\overline{3})(2)(-7)\\
*\,(\overline{10},1)(9) &\to (1,1,1)(6)(9) \oplus (3,1,1)(-4)(9) \oplus (\overline{3},2,1)(1)(9)\\
(10,3)(1) &\to (1,1,3)(-6)(1) \oplus (\overline{3},1,3)(4)(1) \oplus (3,2,3)(-1)(1)\\
(1,\overline{6})(10) &\to (1,1,\overline{6})(0)(10)\\
(\overline{5},\overline{3})(2) &\to (1,2,\overline{3})(3)(2) \oplus (\overline{3},1,\overline{3})(-2)(2)\\
(\overline{15},1)(-6) &\to (1,3,1)(6)(-6) \oplus (\overline{3},2,1)(1)(-6) \oplus (\overline{6},1,1)(-4)(-6)
\end{aligned} \tag{86}
$$

Note that the numbers correspond to transformations according to :

(SU(3) color, SU(2) weak, SU(3) dark)(U(1) Hypercharge)(U'(1) Dark charge)

The SM particle content is given by the 5 and $\overline{10}$ representations, which are displayed with $*$ above, where the terms from $8 \oplus 56$ should be taken twice. We need to account for the all the rest terms by assigning them to corresponding dark sector particles.

Another plausible choice is the $4 \cdot (8) \oplus \overline{28}$. Similar to the Method 1, we can organize the particle content in 1. The content of highlighted rows should be multiplied by 4.

$$
\begin{aligned}
(1) \quad 8 &\to (1,3)(-5) \oplus (5,1)(3)\\
(-4) \quad \overline{28} &\to (1,3)(10) \oplus (\overline{5},\overline{3})(2) \oplus (\overline{10},1)(-6)
\end{aligned} \tag{87}
$$

$$
\begin{aligned}
(1,3)(-5) &\to (1,1,3)(0)(-5)\\
(5,1)(3) &\to (1,2,1)(-3)(3) \oplus (3,1,1)(2)(3)\\
(1,3)(10) &\to (1,1,3)(0)(10)\\
(\overline{5},\overline{3})(2) &\to (1,2,\overline{3})(3)(2) \oplus (\overline{3},1,\overline{3})(-2)(2)\\
(\overline{10},1)(-6) &\to (1,1,1)(6)(-6) \oplus (3,1,1)(-4)(-6) \oplus (\overline{3},2,1)(1)(-6)
\end{aligned} \tag{88}
$$

Rep term	SU(3) color	SU(2) weak	SU(3) dark	U(1) Y_W	U′(1) Y_D	Plausible matter?
$(1,2,1)(-3)(3)$	1	2	1	-3	3	SM Anti-lepton doublet
$(3,1,1)(2)(3)$	3	1	1	2	3	SM d quark
$(3,1,1)(-4)(-6)$	3	1	1	-4	-6	SM u quark
$(1,1,1)(6)(-6)$	1	1	1	6	-6	SM electron
$(\overline{3},2,1)(1)(-6)$	$\overline{3}$	2	1	1	-6	SM anti-quark doublet
$(1,1,3)(0)(-5)$	1	1	3	0	-5	dark quark-1
$(1,1,3)(0)(10)$	1	1	3	0	10	dark quark-2
$(1,2,\overline{3})(3)(2)$	1	2	$\overline{3}$	3	2	Charged weak-dark doublet
$(\overline{3},1,\overline{3})(-2)(2)$	$\overline{3}$	1	$\overline{3}$	-2	2	Charged color-dark triplet

Table 1: The $4 \cdot (8) \oplus \overline{28}$ decomposition

5.3 Spontaneous Symmetry breaking of SU(8) with Cubic interaction

We consider the breaking of SU(8) with Higgs field in adjoint representation, with the inclusion of cubic interaction term. Let's consider the breaking in general case : $\mathrm{SU(8)} \to \mathrm{SU(N)}_1 \otimes \mathrm{SU(N)}_2 \otimes \mathrm{SU(N)}_3$. The potential is :

$$V(\Phi) = \frac{-\mu^2}{2} \cdot \mathrm{Tr}(\Phi^2) + \frac{\lambda_1}{4} \cdot \mathrm{Tr}(\Phi^4) + \frac{\lambda_2}{4} \cdot (\mathrm{Tr}(\Phi^2))^2 + \frac{\lambda_3}{3} \cdot \mathrm{Tr}(\Phi^3) \tag{89}$$

We can consider the diagonal form of field :

$$\Phi = v \cdot \mathrm{diag}(\alpha_1, \cdots, \alpha_N) \tag{90}$$

Imposing

$$\sum \alpha_i = 0 \quad \sum \alpha_i^2 = 1 \tag{91}$$

Substituting into potential we get,

$$V(\Phi) = \frac{-\mu^2}{2} \cdot v^2 + \frac{\lambda_1}{4} \cdot v^4 \sum \alpha_i^4 + \frac{\lambda_2}{4} \cdot v^4 + \frac{\lambda_3}{3} \cdot v^3 \sum \alpha_i^3 \tag{92}$$

We can define coefficients :

$$A = \lambda_1 \sum \alpha_i^4 + \lambda_2 \quad B = \lambda_3 \sum \alpha_i^3 \quad C = -\mu^2 \tag{93}$$

So that Potential is written as :

$$V(\Phi) = A \cdot \frac{v^4}{4} + B \cdot \frac{v^3}{3} + C \cdot \frac{v^2}{2} \tag{94}$$

To obtain the minima condition, we set $\frac{\partial V}{\partial v} = 0$ and omitting the trivial $v = 0$ solution, we obtain Quadratic in v with solution given by :

$$v = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{95}$$

Substituting back into Potential,

$$V = \frac{(B \mp \sqrt{\Delta})^2 \cdot (-5B^2 + 24AC \pm 2B\sqrt{\Delta} + 3\Delta)}{192A^3} \tag{96}$$

where $\Delta = B^2 - 4AC, A > 0, C < 0$. (For a bounded potential with non-trivial minimas)

We impose the conditions $\sum \alpha_i = 0 \quad \sum \alpha_i^2 = 1$ by requiring following :

$$\sum \alpha_i = \sum_{i=1}^{3} N_i v_i = 0 \quad \text{and} \quad \sum \alpha_i^2 = \sum_{i=1}^{3} N_i v_i^2 = 1 \tag{97}$$

we fix $\sum_{i=1}^{3} N_i = 8$ and replace $N_3 \to 8 - N_1 - N_2$ everywhere. The two different cases of solutions are :

Case 1 :

$$\begin{aligned} v_1 &\to \frac{-8N_1v_3 + N_1^2v_3 + N_1N_2v_3 - \sqrt{N_1^2N_2 + N_1N_2^2 - 64N_1N_2v_3^2 + 8N_1^2N_2v_3^2 + 8N_1N_2^2v_3^2}}{N_1^2 + N_1N_2} \\ v_2 &\to \frac{-8v_3 + N_1v_3 + N_2v_3 + \frac{8N_1^2v_3}{N_1^2+N_1N_2} - \frac{N_1^3v_3}{N_1^2+N_1N_2} - \frac{N_1^2N_2v_3}{N_1^2+N_1N_2} + \frac{N_1\sqrt{N_1^2N_2+N_1N_2^2-64N_1N_2v_3^2+8N_1^2N_2v_3^2+8N_1N_2^2v_3^2}}{N_1^2+N_1N_2}}{N_2} \end{aligned} \tag{98}$$

Case 2 :

$$v_1 \rightarrow \frac{-8N_1v_3 + N_1^2v_3 + N_1N_2v_3 + \sqrt{N_1^2N_2 + N_1N_2^2 - 64N_1N_2v_3^2 + 8N_1^2N_2v_3^2 + 8N_1N_2^2v_3^2}}{N_1^2 + N_1N_2}$$
$$v_2 \rightarrow \frac{-8v_3 + N_1v_3 + N_2v_3 + \frac{8N_1^2v_3}{N_1^2+N_1N_2} - \frac{N_1^3v_3}{N_1^2+N_1N_2} - \frac{N_1^2N_2v_3}{N_1^2+N_1N_2} - \frac{N_1\sqrt{N_1^2N_2+N_1N_2^2-64N_1N_2v_3^2+8N_1^2N_2v_3^2+8N_1N_2^2v_3^2}}{N_1^2+N_1N_2}}{N_2} \tag{99}$$

We know we can write :

$$\sum \alpha_i^4 = \sum_{i=1}^{3} N_iv_i^4 = N_1v_1^4 + N_2v_2^4 + (8 - N_1 - N_2)v_3^4 \quad \text{and} \sum \alpha_i^3 = \sum_{i=1}^{3} N_iv_i^3 = v_1 + v_2 + v_3 + 8v_1v_2v_3 \tag{100}$$

With these replacements in place, the value of A (v^4 coefficient) can be written as :
Case 1 :

$$A = \lambda_2 + \lambda_1 \cdot \Big\{(8 - N_1 - N_2)v_3^4 + \frac{\Big(N_1(8 - N_1 - N_2)v_3 + \sqrt{N_1N_2(N_1 + N_2 + 8(-8 + N_1 + N_2)v_3^2)}\Big)^4}{N_1^3(N_1 + N_2)^4} + \frac{\big(-N_2(8 - N_1 - N_2)v_3 + \sqrt{N_1N_2(N_1 + N_2 + 8(-8 + N_1 + N_2)v_3^2)}\big)^4}{N_2^3(N_1 + N_2)^4}\Big\} \tag{101}$$

Case 2 :

$$A = \lambda_2 + \lambda_1 \cdot \Big\{(8 - N_1 - N_2)v_3^4 + \frac{\Big(-N_1(8 - N_1 - N_2)v_3 + \sqrt{N_1N_2(N_1 + N_2 + 8(-8 + N_1 + N_2)v_3^2)}\Big)^4}{N_1^3(N_1 + N_2)^4} + \frac{\big(N_2(8 - N_1 - N_2)v_3 + \sqrt{N_1N_2(N_1 + N_2 + 8(-8 + N_1 + N_2)v_3^2)}\big)^4}{N_2^3(N_1 + N_2)^4}\Big\} \tag{102}$$

We consider the case 1 (equation 98) and the upper sign case in equation 95 :
Including the cubic interaction, the potential with above consideration is :

$$V = \frac{(B - \sqrt{\Delta})^2 \cdot (-5B^2 + 24AC + 2B\sqrt{\Delta} + 3\Delta)}{192A^3} \tag{103}$$

where $\Delta = B^2 - 4AC, A > 0, C < 0$. A and B are given as :

$$A = \lambda_2 + \lambda_1 \cdot \Big\{(8 - N_1 - N_2)v_3^4 + \frac{\Big(N_1(8 - N_1 - N_2)v_3 + \sqrt{N_1N_2(N_1 + N_2 + 8(-8 + N_1 + N_2)v_3^2)}\Big)^4}{N_1^3(N_1 + N_2)^4} + \frac{\big(-N_2(8 - N_1 - N_2)v_3 + \sqrt{N_1N_2(N_1 + N_2 + 8(-8 + N_1 + N_2)v_3^2)}\big)^4}{N_2^3(N_1 + N_2)^4}\Big\} \tag{104}$$

$$B = \frac{\lambda_3}{N_1N_2(N_1 + N_2)^2} \cdot \Big\{N_1^3N_2v_3(3 + 8v_3^2) - N_2\big(N_2 + 8(-8 + N_2)v_3^2\big)\sqrt{N_1N_2(N_1 + N_2 - 8(8 - N_1 - N_2)v_3^2)} + N_1v_3\big((-8 + N_2)N_2(3N_2 + 8v_3^2(-16 + N_2)) - 64v_3\sqrt{N_1N_2(N_1 + N_2 - 8(8 - N_1 - N_2)v_3^2)}\big) + N_1^2\big(2N_2^2v_3(3 + 8v_3^2) + (1 + 8v_3^2)\sqrt{N_1N_2(N_1 + N_2 - 8(8 - N_1 - N_2)v_3^2)} - 24N_2(v_3 + 8v_3^3)\big)\Big\} \tag{105}$$

And $C = -\mu^2$. Substituting, we get expression for potential in terms of : $(v_3, N_1, N_2, \lambda_1, \lambda_2, \lambda_3, C)$. The last 4 are arbitrary constants, and we need to minimize the potential with respect to first 3 parameters. What we shall do is :

- First set the constants value in some appropriate range : $\lambda_1 \in [0,1], \lambda_2 \in [0,1], \frac{\lambda_3^2}{\mu^2} \in [0,1]$.
- Calculate the potential for all possible integer pair values over (N_1, N_2), where $1 \leq N_1 \leq 7$, $1 \leq N_2 \leq 7$, $(N_1 + N_2) \leq 8$. We only run over the pairs such that $N_1 \geq N_2 \geq N_3$. Then, we have to scan over only 9 pairs for each case.
  The cases explicitly are $(3,3,2), (4,3,2), (4,2,2), (4,4,0), (5,2,1), (5,3,0), (6,1,1), (6,2,0), (7,1,0)$
- For each such pair, the potential only depends on v_3 now. To stay in the real range, $v_3 \leq \sqrt{\frac{N_1+N_2}{8(8-N_1-N_2)}}$. We minimize all the potentials with respect to v_3 and note down the minimum value attained by each.
- we compare minimum values of all the $9 \cdot 2 = 18$ potentials and note down the lowest of them all, and say that the corresponding (N_1, N_2) pair has the global minima for given set of $(\lambda_1, \lambda_2, \lambda_3, C)$
- Vary the values $(\lambda_1, \lambda_2, \lambda_3, C)$ to see if we can obtain the minima at some required point.

We perform this in a Google Colab notebook. For simplicity, we consider the case with v_+ solution, but with both cases, to span over all the possible minima values of potential. The Google Colab notebook is publicly available and can be visited by clicking here. A few of the Google Colab notebook outputs from finding the minimum values of potential are given below.

• Set $\lambda_3 = 0$, leading to the no-cubic interaction. We thus expect minima to lie at (4,4,0). The output for $(\lambda_1, \lambda_2, \lambda_3, C) = (0.68, 0.27, 0, -89.90)$ is :

```
        GLobal Minima is obtained at (N1,N2)= ( 4 , 4 ) Case = 1
        GLobal Minima value = -5691.556338028171
        Global Minima No-Cubic Analytical value = -5691.556338028169
        Approximately, Global Minima is also at = ( 4 , 2 ) Case 1 Value at this point
= -5691.556338028168
        Approximately, Global Minima is also at = ( 4 , 3 ) Case 1 Value at this point
= -5691.556338028164
        Approximately, Global Minima is also at = ( 4 , 4 ) Case 1 Value at this point
= -5691.556338028171
        Approximately, Global Minima is also at = ( 4 , 2 ) Case 2 Value at this point
= -5691.556338028171
        Approximately, Global Minima is also at = ( 4 , 3 ) Case 2 Value at this point
= -5691.556338028171
        Approximately, Global Minima is also at = ( 4 , 4 ) Case 2 Value at this point
= -5691.556338028171
```

We observe that the global minima is achieved at (4,4,0) and it agrees with the analytical value of global minima in the case without the cubic interaction. We also see that we have degenerate sets of minima at (4,2,2) and (4,3,1).

• Set $\lambda_3 \neq 0$, leading to the cubic interaction. We thus want to investigate the value and location of the minima for different sets of parameters. The output for $(\lambda_1, \lambda_2, \lambda_3, C) = (0.68, 0.27, 3.2, -20.90)$ is :

```
GLobal Minima is obtained at (N1,N2)= ( 6 , 2 ) Case = 2
GLobal Minima value = -413.40967760465975
Global Minima No-Cubic Analytical value = -307.61267605633793
Approximately, Global Minima is also at = ( 4 , 2 ) Case 2 Value at this point
= -413.40967760465753
Approximately, Global Minima is also at = ( 5 , 2 ) Case 2 Value at this point
= -413.4096776046497
Approximately, Global Minima is also at = ( 6 , 1 ) Case 2 Value at this point
= -413.40967760465867
Approximately, Global Minima is also at = ( 6 , 2 ) Case 2 Value at this point
= -413.40967760465975
```

We see that the minima value with cubic interaction is lower than the expected analytic result for the no-cubic interaction. We see the location of the minima is at (6,2,0). We can try to vary the parameters such that we may obtain the minima at (3,3,2) or (5,3,0), which would be more interesting in regards to this project.

• Set $\lambda_3 \neq 0$, leading to the cubic interaction. We thus want to investigate the value and location of the minima for different sets of parameters. The output for $(\lambda_1, \lambda_2, \lambda_3, C) = (0.68, 0.27, 3.20, -89.90)$ is :

```
GLobal Minima is obtained at (N1,N2)= ( 5 , 3 ) Case = 2
GLobal Minima value = -6121.763940068969
Global Minima No-Cubic Analytical value = -5691.556338028169
Approximately, Global Minima is also at = ( 3 , 3 ) Case 1 Value at this point
= -6121.763940068965
Approximately, Global Minima is also at = ( 3 , 3 ) Case 2 Value at this point
= -6121.763940068968
Approximately, Global Minima is also at = ( 4 , 3 ) Case 2 Value at this point
= -6121.763940068963
Approximately, Global Minima is also at = ( 5 , 2 ) Case 2 Value at this point
= -6121.763940068942
Approximately, Global Minima is also at = ( 5 , 3 ) Case 2 Value at this point
= -6121.763940068969
```

We can see that we obtain the location of the minima at (5,3,0). We can see that the degenerate minimas also contain the minima at (3,3,2).

By tuning these parameters, checking the scale invariance of the minima location, we can convince that the inclusion of the cubic interaction causes the potential to have global minima at the locations (6,2,0), (5,3,0) and even (3,3,0), depending upon the values of parameters $(\lambda_1, \lambda_2, \lambda_3, C)$. Therefore, the spontaneous symmetry breaking of SU(8) to SU(5) $\otimes$ SU(3) $\otimes$ U(1) or to SU(3) $\otimes$ SU(3) $\otimes$ SU(2) $\otimes$ U(1 should be possible with the inclusion of cubic interaction in the potential for the adjoint representation.

6 Conclusion and Outlook

In this undergradute thesis project, we have established an idea for unifying the symmetry group of dark matter interactions with the gauge group of Standard Model. We chose the symmetry group for the dark matter interactions to be SU(3), for a asymptotic free dynamics like the QCD. We conducted a review study on the established Standard Model of Particle Physics and Georgi-Glashow SU(5) Model, to take up some insights for 'Unifying the Dark QCD'.

We also went through the various Group Theoretical techniques required for Grand Unified Theories Model building. A short and concise review of Young Tableaux, Dynkin Diagrams, Anomaly coefficients and the Symmetry breaking mechanism have been presented. An interested reader can use these as a starting point to move forward to the advanced resources that have been mentioned.

Finally, we explore the possibilities and mathematical aspects of SU(8) to be a Unifying model for the Dark matter Symmetry group and Standard Model Gauge group. We conduct an original study of the symmetry breaking of SU(8) in adjoint representation with inclusion of the Cubic interaction. With the empirical evidence presented in section 5.3, we conclude that the symmetry breaking of SU(8) to $SU(5) \otimes SU(3) \otimes U(1)$ or to $SU(3) \otimes SU(3) \otimes SU(2) \otimes U(1)$ should be possible with the inclusion of cubic interaction in the potential for the adjoint representation.

The future directions for the project include the calculation of the Yukawa sector and embedding the necessary Higgs fields in the representations of the SU(8). Another direction can be exploring the SU(7) group as the candidate group and repeating the similar procedure, accordingly[11].

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[11] $SU(7)$ breaking directly to $SU(3) \otimes SU(3) \otimes SU(2) \otimes U(1)$ can be explored.

References

[1] G. Aad *et al.* [ATLAS], Phys. Lett. B **716**, 1-29 (2012) doi:10.1016/j.physletb.2012.08.020 [arXiv:1207.7214 [hep-ex]].

[2] S. Chatrchyan *et al.* [CMS], Phys. Lett. B **716**, 30-61 (2012) doi:10.1016/j.physletb.2012.08.021 [arXiv:1207.7235 [hep-ex]].

[3] N. Aghanim *et al.* [Planck], Astron. Astrophys. **641**, A6 (2020) [erratum: Astron. Astrophys. **652**, C4 (2021)] doi:10.1051/0004-6361/201833910 [arXiv:1807.06209 [astro-ph.CO]].

[4] R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. **38**, 1440-1443 (1977) doi:10.1103/PhysRevLett.38.1440

[5] S. Weinberg, Phys. Rev. Lett. **40**, 223-226 (1978) doi:10.1103/PhysRevLett.40.223

[6] J. L. Feng, Ann. Rev. Astron. Astrophys. **48**, 495-545 (2010) doi:10.1146/annurev-astro-082708-101659 [arXiv:1003.0904 [astro-ph.CO]].

[7] Y. Bai and P. Schwaller, Phys. Rev. D **89**, no.6, 063522 (2014) doi:10.1103/PhysRevD.89.063522 [arXiv:1306.4676 [hep-ph]].

[8] W. de Boer, Prog. Part. Nucl. Phys. **33**, 201-302 (1994) doi:10.1016/0146-6410(94)90045-0 [arXiv:hep-ph/9402266 [hep-ph]].

[9] H. Georgi, doi:10.1201/9780429499210

[10] H. Georgi and S. L. Glashow, Phys. Rev. Lett. **32**, 438-441 (1974) doi:10.1103/PhysRevLett.32.438

[11] P. W. Higgs, Phys. Rev. Lett. **13**, 508-509 (1964) doi:10.1103/PhysRevLett.13.508

[12] F. Englert and R. Brout, Phys. Rev. Lett. **13**, 321-323 (1964) doi:10.1103/PhysRevLett.13.321

[13] G. S. Guralnik, C. R. Hagen and T. W. B. Kibble, Phys. Rev. Lett. **13**, 585-587 (1964) doi:10.1103/PhysRevLett.13.585

[14] S. L. Glashow, Nucl. Phys. **22**, 579-588 (1961) doi:10.1016/0029-5582(61)90469-2

[15] S. Weinberg, Phys. Rev. Lett. **19**, 1264-1266 (1967) doi:10.1103/PhysRevLett.19.1264

[16] A. Salam, Conf. Proc. C **680519**, 367-377 (1968) doi:10.1142/9789812795915_0034

[17] S. L. Glashow, J. Iliopoulos and L. Maiani, Phys. Rev. D **2**, 1285-1292 (1970) doi:10.1103/PhysRevD.2.1285

[18] R. Slansky, Phys. Rept. **79**, 1-128 (1981) doi:10.1016/0370-1573(81)90092-2

[19] L. F. Li, Phys. Rev. D **9**, 1723-1739 (1974) doi:10.1103/PhysRevD.9.1723

[20] J. Thole, "Family symmetry Grand Unified Theories" (2019) : https://fse.studenttheses.ub.rug.nl/19026/1/ThesisFinalJelleThole.pdf

[21] A. Greljo "Towards unification: SU(5) and SO(10)" (2012) : http://www-f1.ijs.si/ ziherl/Greljo12.pdf

[22] M. Romao "The SU(5) Grand Unification Theory Revisited" (2011) : https://fenix.tecnico.ulisboa.pt/downloadFile/395143154268/Thesis.pdf

[23] A. Pich, [arXiv:1201.0537 [hep-ph]].

[24] I. Nisandzic "SU(5) and SO(10) Unification" (2011) : http://www-f1.ijs.si/ ziherl/Nisandzic11.pdf

[25] U. Sarkar, Int. J. Mod. Phys. A **22**, 931-936 (2007) doi:10.1142/S0217751X07034969 [arXiv:hep-ph/0606206 [hep-ph]].

[26] R. Feger and T. W. Kephart, Comput. Phys. Commun. **192**, 166-195 (2015) doi:10.1016/j.cpc.2014.12.023 [arXiv:1206.6379 [math-ph]].

[27] R. Feger, T. W. Kephart and R. J. Saskowski, Comput. Phys. Commun. **257**, 107490 (2020) doi:10.1016/j.cpc.2020.107490 [arXiv:1912.10969 [hep-th]].

[28] H. F. Jones, Groups, Representations and Physics (Taylor and Francis Ltd, 1998)

[29] M. E. Peskin and D. V. Schroeder, "An Introduction to quantum field theory,"