Collider Phenomenology Lectures

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1 Exercise Solutions

• Exercise 1.1 :

We know the Reaction Rate R is the given by the product of Luminosity L and cross-section σ for the process :

$$R = L \cdot \sigma$$

The Luminosity is given as $L = 10^{34} cm^{-2} s^{-1}$. We obtain the net cross-section by adding up the partial cross-sections as follows :

$$\sigma = (54.67 + 4.278 + 1.513 + 0.986 + 0.614 + 0.553)\ pb$$
$$\Rightarrow \quad \sigma = 62.614\ pb$$

Therefore, Reaction rate is given by :

$$R = 62.614 \times 10^{-24} \times 10^{-12} \times 10^{34}\ s^{-1}$$

$$\Rightarrow \quad R = 0.62614\ s^{-1}$$

Therefore 0.62614 Higgs bosons are produced per second at the LHC. (one Higgs event every two seconds)
Multiplying with 365×86400 we get the production over a year,
Producing approximately 1.97×10^7 Higgs bosons a year.

• Exercise 1.2 :

We consider the process $gg \to t\bar{t}H$ in the Center of Mass frame with Momentum parametrization as follows :

$$p_1 = \frac{\sqrt{\hat{s}}}{2}(1, 0, 0, 1)$$
$$p_2 = \frac{\sqrt{\hat{s}}}{2}(1, 0, 0, -1)$$
$$p_H = (E_H, 0, \sqrt{E_H^2 - m_H^2}\sin\theta_H, \sqrt{E_H^2 - m_H^2}\cos\theta_H)$$

We write the process $p_1 + p_2 \to p_t + p_{\bar{t}} + p_H$ as $p_1 + p_2 \to P_{t\bar{t}} + p_H$ such that $P_{t\bar{t}} = p_t + p_{\bar{t}}$.

Here we can observe $\hat{s} = (p_t + p_{\bar{t}} + p_H)^2$ will be due to 4-momentum conservation which will be imposed through the delta function later on. Similaly we define, $s_{t\bar{t}} = P_{t\bar{t}}^2 = (p_t + p_{\bar{t}})^2$

Let us write the $2 \to 3$ Phase space volume element for the given scattering process. We will use the general formula for the phase space element. For simplicity, We substitute $n = 3$ and $D = 4$ to get :

$$d\Phi_{t\bar{t}H} = \frac{1}{(2\pi)^5}\Bigg\{ d^4p_t\ \delta(E_t^2 - |\overrightarrow{p_t}|^2 - m_t^2)\ \Theta(E_t)\ d^4p_{\bar{t}}\ \delta(E_{\bar{t}}^2 - |\overrightarrow{p_{\bar{t}}}|^2 - m_t^2)\ \Theta(E_{\bar{t}})$$
$$\times\ d^4p_H\ \delta(E_H^2 - |\overrightarrow{p_H}|^2 - m_H^2)\ \Theta(E_H)\Bigg\} \cdot \delta(p_1 + p_2 - p_t - p_{\bar{t}} - p_H)$$

Now we will introduce some factors to impose the conditions that we chose.
Let's write the delta function in the following manner to impose the condition $P_{t\bar{t}} = p_t + p_{\bar{t}}$.

$$\delta(p_1 + p_2 - p_t - p_{\bar{t}} - p_H) = d^4 P_{t\bar{t}}\ \delta(P_{t\bar{t}} - p_t - p_{\bar{t}}) \cdot \delta(p_1 + p_2 - P_{t\bar{t}} - p_H)$$

Similalry, to impose the condition $s_{t\bar{t}} = P_{t\bar{t}}^2$ we will introduce the following factor of unity into the expression.For consistency, we will have the positive energies for the pseudo state only.

$$ds_{t\bar{t}}\ \delta(s_{t\bar{t}} - P_{t\bar{t}}^2)\ \Theta(E_{t\bar{t}})$$

Inserting these factors into the Original expression and rearranging we get,

$$\begin{aligned} d\Phi_{t\bar{t}H} = \frac{ds_{t\bar{t}}}{(2\pi)^5} & \left\{ d^4 p_t\ \delta(E_t^2 - |\overrightarrow{p_t}|^2 - m_t^2)\ \Theta(E_t)\ d^4 p_{\bar{t}}\ \delta(E_{\bar{t}}^2 - |\overrightarrow{p_{\bar{t}}}|^2 - m_t^2)\ \Theta(E_{\bar{t}})\ \delta(P_{t\bar{t}} - p_t - p_{\bar{t}}) \right\} \\ \times & \left\{ d^4 p_H\ \delta(E_H^2 - |\overrightarrow{p_H}|^2 - m_H^2)\ \Theta(E_H)\ d^4 P_{t\bar{t}}\ \delta(P_{t\bar{t}}^2 - s_{t\bar{t}})\ \Theta(E_{t\bar{t}})\ \delta(p_1 + p_2 - P_{t\bar{t}} - p_H) \right\} \end{aligned}$$

Now we can observe that the $2 \to 3$ Phase space element can be written as a combination of two $2 \to 2$ Phase space elements considering a pseudo state as follows :

$$d\Phi_{t\bar{t}H} = \frac{ds_{t\bar{t}}}{(2\pi)} \cdot d\Phi_2(P_{t\bar{t}} \to p_t + p\bar{t}) \cdot d\Phi_2(p_1 + p_2 \to P_{t\bar{t}} + p_H)$$

Now We follow the standard procedure for solving a $2 \to 2$ Phase space element and upon replacing with the appropriate pseudo state parameters we get :

$$d\Phi_2(P_{t\bar{t}} \to p_t + p_{\bar{t}}) = \frac{1}{16\pi^2} \cdot \frac{\sqrt{\lambda(s_{t\bar{t}}, m_t^2, m_t^2)}}{2 \cdot s_{t\bar{t}}} \cdot d\Omega_{t\bar{t}}$$

Along with,

$$d\Phi_2(p_1 + p_2 \to P_{t\bar{t}} + p_H) = \frac{1}{16\pi} \cdot \frac{\sqrt{\lambda(\hat{s}, s_{t\bar{t}}, m_H^2)}}{\hat{s}} \cdot \sin\theta_H\ d\theta_H$$

Combining the factors we get,

$$d\Phi_{t\bar{t}H} = \frac{1}{2^{10}\pi^4} \frac{\sqrt{\lambda(\hat{s}, s_{t\bar{t}}, m_H^2)}}{\hat{s}} \frac{\sqrt{\lambda(s_{t\bar{t}}, m_t^2, m_t^2)}}{s_{t\bar{t}}} ds_{t\bar{t}}\ d\Omega_{t\bar{t}}\ \sin\theta_H\ d\theta_H$$

The minimum energy requirements for the 3 corresponding processes are :

$$\sqrt{\hat{s}} \geq 2m_t + m_H, \quad \sqrt{s_{t\bar{t}}} \geq 2m_t, \quad \sqrt{\hat{s}} \geq \sqrt{s_{t\bar{t}}} + m_H$$

We introduce the corresponding Θ functions to account for the minimum energy requirements and we finally get our expression as :

$$\begin{aligned} d\Phi_{t\bar{t}H} = \frac{1}{2^{10}\pi^4} & \frac{\sqrt{\lambda(\hat{s}, s_{t\bar{t}}, m_H^2)}}{\hat{s}} \frac{\sqrt{\lambda(s_{t\bar{t}}, m_t^2, m_t^2)}}{s_{t\bar{t}}} ds_{t\bar{t}}\ d\Omega_{t\bar{t}}\ \sin\theta_H\ d\theta_H \\ & \times \Theta(\sqrt{\hat{s}} - 2m_t - m_H)\ \Theta(s_{t\bar{t}} - 4m_t^2)\ \Theta([\sqrt{\hat{s}} - m_H]^2 - s_{t\bar{t}}) \end{aligned}$$

• Exercise 2.2 :

We know the $SU(N)$ group can be represented by Unitary matrices with unit determinant.

$$SU(N) = \{U \in \mathbb{C}^{N\times N} |\ UU^{\dagger} = 1 \wedge det\ U = 1\}$$

where each $U \in SU(N)$ can be written as :

$$U = \exp\{\ i\theta_a T^a\} \quad : \theta_a \in \mathbb{R}$$

such that $\{T^a\}$ are Traceless Hermitian Operators.

Therefore, the dimension of $SU(N)$ will be given by the number of Independent Traceless-Hermitian Operators spanning the space of $SU(N)$.

Consider the most general $N \times N$ Matrix (Operator) with complex entries. Each entry will be characterized by 2 constants $(a_j + ib_j)$. Therefore, the Operator can be characterized by $2N^2$ Independent constants.

For the Operator to be Hermitian, The Diagonal entries must be real. Therefore, N number of constants corresponding to the imaginary parts of diagonal entries are all zero. Thus, the number of Independent constants is reduced by N.

Now the number of Independent constants corresponding to Non-diagonal entries is $2 \cdot (N^2 - N)$. Because of Hermicity, these entries appear in complex conjugate pairs and thus the total number of independent constants is reduced by $\frac{2\cdot(N^2-N)}{2} = (N^2 - N)$.

Applying the Traceless condition, we impose a relation amongst the diagonal elements and further reduce the total number of Independent constants by 1.

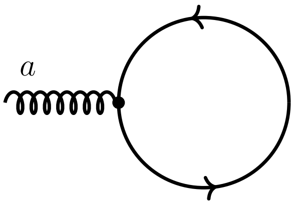
Final number of total Independent constants $= 2N^2 - N - (N^2 - N) - 1 = N^2 - 1$.

Therefore, A Traceless-Hermitian Operator can be characterized by $(N^2 - 1)$ constants.

As a result, corresponding to these constants, we can construct $(N^2 - 1)$ number of Independent Traceless-Hermitian Operators which will span the space of $SU(N)$ as specified. Therefore, dimension of $SU(N)$ is $(N^2 - 1)$.

• Exercise 3.1 :

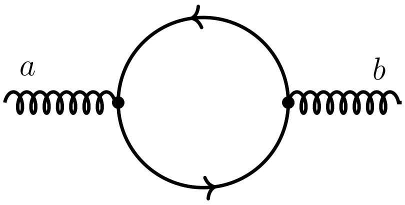
1)



We, know physically color factor for this diagram should be zero as a gluon cannot be exchanged between quarks of same color. Mathematically it can be given by t^a_{ij} where we set $j = i$ and sum over i.

$$\Rightarrow \quad t^a_{ii} = Tr(T^a) = 0$$

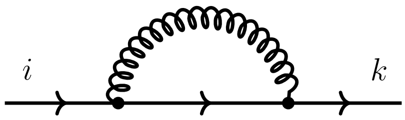
2)



Color Factor for the above diagram can be given by combination of t^a_{ij} & t^b_{kl}. We will set $k = j$ & $l = i$. Therefore color factor will be $t^a_{ij}t^b_{ji}$ summed over i & j.

$$\Rightarrow \quad t^a_{ij}t^b_{ji} = Tr(T^aT^b) = T_R\ \delta^{ab}$$

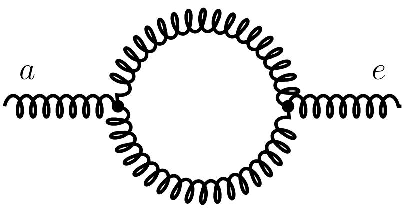
3)



Similarly, Color factor for above diagram can also be given by combinations of t^a_{ji} & t^b_{kl}. Here, we set $l = j$ & $b = a$. Therefore, the color factor will be $t^a_{kj}t^a_{ji}$ summed over j & a. This is just the Casimir Identity and therefore :

$$\Rightarrow \quad t^a_{kj}t^a_{ji} = C_F\ \delta_{ki}$$

4)



We use combination of 3-gluon vertex color factors for the above diagram. We use $(F^a)_{bc}$ & $(F^e)_{fg}$. Here, we set $f = c$ & $g = b$ and sum over b & c.

$$\Rightarrow \quad (F^a)_{bc}(F^e)_{cb} = Tr(F^aF^e) = -f^{abc}f^{ecb}$$

We know $(F^a)_{bc} = -if^{abc}$ where f^{abc} is given as follows :

$$f^{abc} = \frac{1}{i\,T_R}\Big[Tr(T^aT^bT^c) - Tr(T^aT^cT^b)\Big]$$

We can expand it as follows :

$$f^{abc} = \frac{1}{i\,T_R}\Big[t^a_{ij}t^b_{jk}t^c_{ki} - t^b_{ij}t^a_{jk}t^c_{ki}\Big]$$

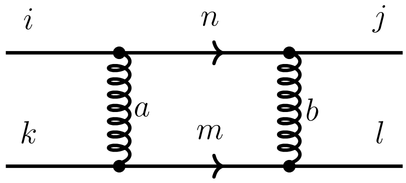
And use the following Identity :

$$t^a_{ij}t^a_{kl} = T_R\Big(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\Big)$$

Therefore, expanding and simplifying further we get the following result :

$$\Rightarrow \quad (F^a)_{bc}(F^e)_{cb} = Tr(F^aF^e) = -f^{abc}f^{ecb} = 2\;T_RN_c\;\delta^{ae} = C_A\delta^{ae}$$

5)



For the above diagram, we use 4 combinations of gluon-quark vertex. Following similar procedure as we did before and setting up appropriate variables to be summed over, we will get :

$$\Rightarrow \quad t^a_{mk}t^a_{ni}t^b_{lm}t^b_{jn}$$

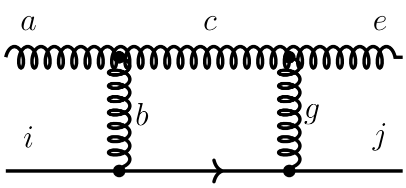
Using the above mentioned identities to sum over a, b & then summing over m, n we get the following relation :

$$\Rightarrow \quad (T_R)^2\Big[\Big(1 + \frac{1}{N_c^2}\Big)\delta_{ji}\delta_{lk} - \frac{2}{N_c}\delta_{jk}\delta_{li}\Big]$$

If $i = j$ & $k = l$ (as indicated in the question) we get the following simplified result :

$$\Rightarrow \quad (T_R)^2(N_c^2 - 1) = \frac{C_AC_F}{2}$$

6)



Again, using combination of the quark-gluon and gluon-gluon vertices, we can write the Color factor for the above diagram as follows :

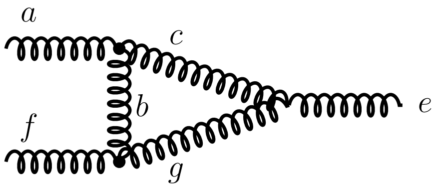
$$\Rightarrow \quad (F^a)_{bc}(F^e)_{cg}t^g_{jk}t^b_{ki}$$

If $a = e$ & $i = j$ (as indicated in the question) we get the following simplified result :

$$\Rightarrow \quad C_AC_FN_c = 2(T_R)^2N_c(N_c^2 - 1) = C_A^2C_F$$

We used the identities : $(F^a)_{bc}(F^a)_{cg} = C_A\delta_{bg}$ & $t^b_{jk}t^b_{ki} = C_F\delta_{ji}$ & $\delta_{ii} = N_c$.

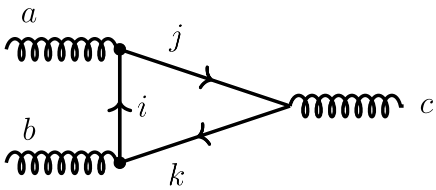
7)



Using the combination of 3 Gluon vertex, we can write the Color factor as :

$$\Rightarrow \quad (F^a)_{bc}(F^e)_{cg}(F^f)_{gb} = Tr(F^a F^e F^f)$$

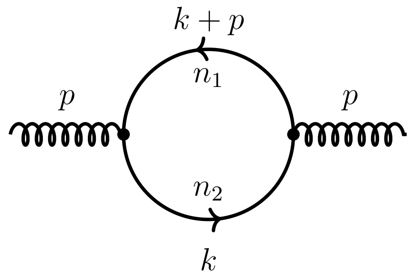
8)

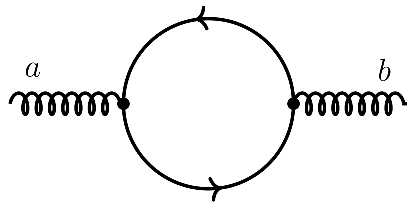


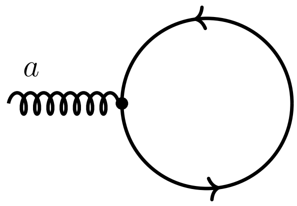
Using combination of quark-gluon vertices, we can write :

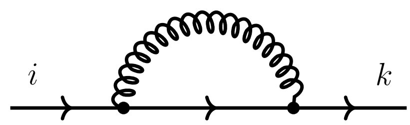
$$\Rightarrow \quad t^a_{ji} t^b_{ik} t^c_{kj} = Tr(T^a T^b T^c)$$

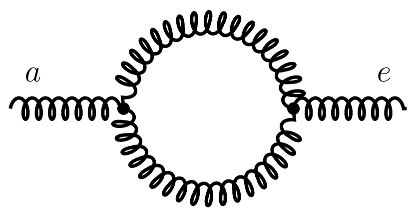
- **Diagrams :**

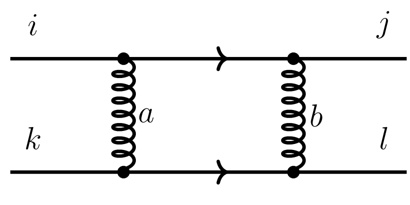


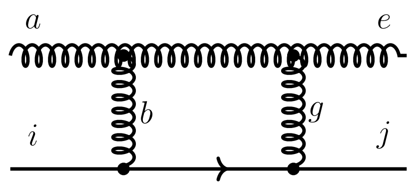




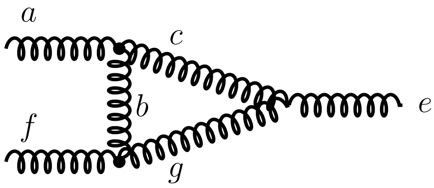




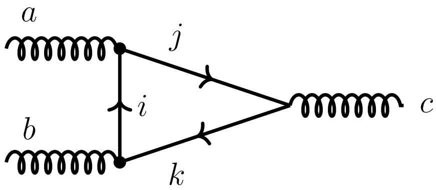




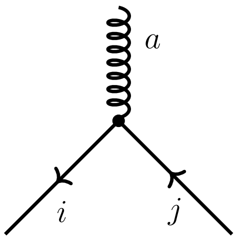
7)



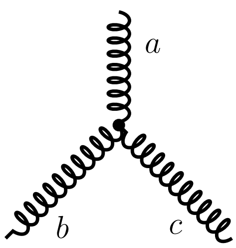
8)



Diagrams for Convention :



$\Rightarrow \; t^a_{ij}$



$\Rightarrow \; (F^a)_{bc} = -if^{abc}$

• Exercise 4.1 :

We use the Identity :

$$t^a_{ij} t^a_{kl} = T_R\Big(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\Big)$$

to expand as follows :

$$Tr(t^a t^b t^a t^b) = t^a_{ij} t^b_{jk} t^a_{kl} t^b_{li}$$

$$\Rightarrow \quad = T_R\Big(\delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\Big) \cdot T_R\Big(\delta_{ji}\delta_{lk} - \frac{1}{N_c}\delta_{jk}\delta_{li}\Big)$$

$$\Rightarrow \quad = (T_R)^2\Big(N_c - \frac{2}{N_c}(N_c)^2 + \frac{1}{N_c}\Big)$$

$$\Rightarrow \quad Tr(t^a t^b t^a t^b) = -(T_R)^2\Big(\frac{N_c^2 - 1}{N_c}\Big)$$

Similarly, (Now we use the Casimir Identity)

$$Tr(t^a t^b t^b t^a) = t^a_{ij} t^b_{jk} t^b_{kl} t^a_{li}$$

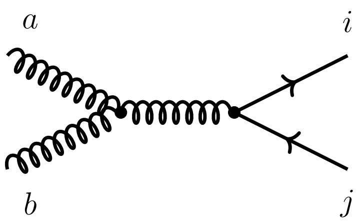
$$\Rightarrow \quad = t^a_{ij}(C_F \delta_{jl}) t^a_{li}$$

$$\Rightarrow \quad = C_F t^a_{ij} t^a_{ji} = (C_F)^2 N_c$$

$$\Rightarrow \quad Tr(t^a t^b t^b t^a) = (T_R)^2 \frac{(N_c^2 - 1)^2}{N_c}$$

Let us Calculate the Color Factors for the following Diagrams :

a)



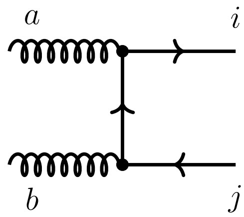
Color factor for the above Diagram can be given by $(F^c)_{ab}\, t^c_{ij}$ summed over c. Using the following Identity :

$$T^a T^b - T^b T^a = i f^{abc} T^c$$

we get the Color Factor for Diagram (a) as :

$$\Rightarrow \quad C^{(a)} = (T^b T^a - T^a T^b)_{ij}$$

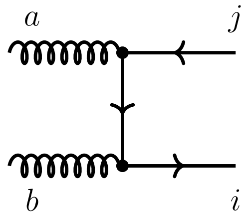
b)



Color Factor for the diagram (b) can be directly given by :

$$\Rightarrow \quad C^{(b)} = (T^b T^a)_{ij}$$

c)



Color Factor for the diagram (c) can be directly given by :

$$\Rightarrow \quad C^{(c)} = (T^a T^b)_{ij}$$

Now we can calculate all the combinations as follows :

$$C^{(a)}C^{(a)\dagger} = (T^bT^a - T^aT^b)_{ij}(T^aT^b - T^bT^a)_{ji}$$
$$\Rightarrow \quad C^{(a)}C^{(a)\dagger} = 2\ [Tr(T^aT^bT^bT^a) - Tr(T^aT^bT^aT^b)]$$
$$C^{(a)}C^{(b)\dagger} = (T^bT^a - T^aT^b)_{ij}(T^aT^b)_{ji}$$
$$\Rightarrow \quad C^{(a)}C^{(b)\dagger} = [Tr(T^aT^bT^bT^a) - Tr(T^aT^bT^aT^b)]$$
$$C^{(b)}C^{(a)\dagger} = (T^bT^a)_{ij}(T^aT^b - T^bT^a)_{ji}$$
$$\Rightarrow \quad C^{(b)}C^{(a)\dagger} = [Tr(T^aT^bT^bT^a) - Tr(T^aT^bT^aT^b)]$$

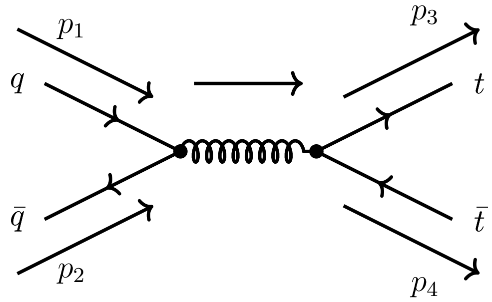
Similarly we can calculate,

$$\Rightarrow \quad C^{(c)}C^{(a)\dagger} = C^{(a)}C^{(c)\dagger} = -[Tr(T^aT^bT^bT^a) - Tr(T^aT^bT^aT^b)]$$

$$\Rightarrow \quad C^{(c)}C^{(c)\dagger} = C^{(b)}C^{(b)\dagger} = Tr(T^aT^bT^bT^a)$$
$$\Rightarrow \quad C^{(c)}C^{(b)\dagger} = C^{(b)}C^{(c)\dagger} = Tr(T^aT^bT^aT^b)$$

The combinations $C^{(c)}C^{(b)\dagger}$ & $C^{(b)}C^{(c)\dagger}$ can be dropped in the Leading order calculation.

• Exercise 4.2 :



Assumption : Because the quark masses are negligible compared with the top quark mass, we assume the initial state quarks as massless. Therefore, the momentum parametrisation in Center Of Mass Frame will be as follows : ($E_{cm} = 2E$)

$$p_1 = (E, 0, 0, E)$$
$$p_2 = (E, 0, 0, -E)$$
$$p_3 = (E, \vec{k})$$
$$p_4 = (E, -\vec{k})$$

Such that, $p_1^2 = p_2^2 = 0$ & $p_3^2 = p_4^2 = m_t^2 = E^2 - |\vec{k}|^2$, where $\vec{k} \cdot \hat{z} = |\vec{k}| \cos\theta$.

Now, for our $p_1 + p_2 \to p_3 + p_4$ reaction we know the differential Cross-section is given by :

$$d\sigma = \frac{J}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \cdot |\mathcal{M}|^2 \cdot d\Phi_{2\to 2}$$

No Identical particles in final state, $\Rightarrow J = 1$. Substituting we get,

$$d\sigma = \frac{1}{8E^2} \cdot |\mathcal{M}|^2 \cdot d\Phi_{2\to 2}$$

Now, Let us focus on the Matrix Element first. (We use the Feynman gauge where $\lambda = 1$). Following the Feynman rules, it can be given as :

$$\mathcal{M} = \Big[\bar{v}(p_2, s_2)\Big(-ig_s(t^a_{ij})\gamma^\mu\Big)u(p_1, s_1)\Big(\frac{-i\delta^{ab} g_{\mu\nu}}{(p_1+p_2)^2}\Big)\bar{u}(p_3, s_3)\Big(-ig_s(t^b_{kl})\gamma^\nu\Big)v(p_4, s_4)\Big]$$

$$\Rightarrow \quad \mathcal{M} = \frac{ig_s^2(t^a_{ij})(t^a_{kl})}{4E^2}\Big[\bar{v}(p_2, s_2)\gamma^\mu u(p_1, s_1)\bar{u}(p_3, s_3)\gamma_\mu v(p_4, s_4)\Big]$$

To find the $|\mathcal{M}|^2$, we first separately calculate the color contribution, (sum over colors of all quarks, average over the initial quarks, thus additional N_c^2 factor in the denominator)

$$\Rightarrow \quad \Big(t^a_{ij}t^a_{kl}\Big) \cdot \Big(t^a_{ij}t^a_{kl}\Big)^*$$
$$\Rightarrow \quad \Big(t^a_{ij}t^a_{ji}\Big) \cdot \Big(t^a_{kl}t^a_{lk}\Big)$$
$$\Rightarrow \quad \Big(T_R(N_c^2 - 1)\Big)^2$$

$$\Rightarrow \frac{|\mathcal{M}|^2}{N_c^2} = \frac{g_s^4 T_R^2 (N_c^2-1)^2}{16E^4(N_c^2)}\Big[\bar{v}(p_2, s_2)\gamma^\mu u(p_1, s_1)\bar{u}(p_1, s_1)\gamma^\nu v(p_2, s_2)\Big]\Big[\bar{u}(p_3, s_3)\gamma_\mu v(p_4, s_4)\bar{v}(p_4, s_4)\gamma_\nu u(p_3, s_3)\Big]$$

Now we sum over spins of all quarks, and average over the initial quarks. Therefore additional factor of 1/4 is introduced.

$$\Rightarrow \frac{1}{4\cdot N_c^2}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = \frac{g_s^4 T_R^2 (N_c^2-1)^2}{16E^4(4N_c^2)}\Big[Tr(\not{p}_2\gamma^\mu \not{p}_1\gamma^\nu)\Big]\Big[Tr((\not{p}_3+m_t)\gamma_\mu(\not{p}_4-m_t)\gamma_\nu)\Big]$$

where we have used the Identities :

$$\Rightarrow \sum_s u(p,s)\bar{u}(p,s) = \not{p}+m \qquad \Rightarrow \sum_s v(p,s)\bar{v}(p,s) = \not{p}-m$$

Solving the Traces using Gamma matrices properties and upon simplification, we will get :

$$\Rightarrow \frac{1}{4\cdot N_c^2}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = \frac{g_s^4 T_R^2 (N_c^2-1)^2}{16E^4(4N_c^2)}\Big\{32\Big[(p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)+m_t^2(p_1\cdot p_2)\Big]\Big\}$$

Substituting and solving we get,

$$\Rightarrow \frac{1}{4\cdot N_c^2}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = \frac{g_s^4 T_R^2 (N_c^2-1)^2}{N_c^2}\Big[\Big(1+\frac{m_t^2}{E^2}\Big)+\Big(1-\frac{m_t^2}{E^2}\Big)\cdot\cos^2\theta\Big]$$

We know the Mandelstam variable t is given by $t=(p_1-p_3)^2$. Substituting and solving, we get relation between $\cos\theta$ & t as follows :

$$\Rightarrow \quad \Big(\sqrt{1-\frac{m_t^2}{E^2}}\Big)\cdot\cos\theta = \frac{t+2E^2-m_t^2}{2E^2}$$

Substituting this relation, we get the required Amplitude squared in terms of t as follows :

$$\Rightarrow \quad |\mathcal{M}|^2 = \frac{g_s^4\ T_R^2(N_c^2-1)^2}{N_c^2}\Big[2+\frac{t}{E^2}+\Big(\frac{t-m_t^2}{2E^2}\Big)^2\Big]$$

Now, we know the Phase space element is given by : ($E_{cm}=2E$)

$$\Rightarrow \quad d\Phi_{2\to 2} = \frac{1}{16\pi}\frac{\sqrt{\lambda(4E^2,m_t^2,m_t^2)}}{4E^2}\cdot d\cos\theta$$

In terms of t it can be given as :

$$\Rightarrow \quad d\Phi_{2\to 2} = \frac{1}{16\pi}\frac{\sqrt{\lambda(4E^2,m_t^2,m_t^2)}}{4E^2}\cdot\frac{1}{2E\sqrt{E^2-m_t^2}}\cdot dt$$

Now we can substitue everything back into our original formula :

$$d\sigma = \frac{1}{8E^2}\cdot|\mathcal{M}|^2\cdot d\Phi_{2\to 2}$$

$$d\sigma = \frac{g_s^4}{2^{10}\pi}\cdot\frac{T_R^2(N_c^2-1)^2}{N_c^2}\cdot\frac{\sqrt{\lambda(4E^2,m_t^2,m_t^2)}}{E^5\cdot\sqrt{E^2-m_t^2}}\cdot\Big[2+\frac{t}{E^2}+\Big(\frac{t-m_t^2}{2E^2}\Big)^2\Big]\cdot dt$$

• Exercise 5.2 :

Physical polarisation sum for massless vector bosons is given as :

$$d^{\mu\nu} = \sum_{\lambda\ phys} \epsilon^{\mu}_{\lambda}(p)\epsilon^{\nu}_{\lambda}(p)$$

We can express any general rank 2 Tensor depending on (p^{μ}, n^{μ}) as follows :

$$d^{\mu\nu}(p,n) = a \cdot g^{\mu\nu} + b \cdot p^{\mu}n^{\nu} + c \cdot p^{\nu}n^{\mu} + d \cdot p^{\mu}p^{\nu} + e \cdot n^{\mu}n^{\nu}$$

Now along with $p^2 = 0, n^2 = 0$ we impose the required conditions one by one. Imposing $p_{\mu}d^{\mu\nu} = 0$ we get an expression of following form :

$$p^{\nu}\Big(a + c(p \cdot n)\Big) + n^{\nu}\Big(e(p \cdot n)\Big) = 0$$

Now because we require $p \cdot n \neq 0$, p & n must be linearly independent. Therefore, the coefficients in the above expression must be zero.

$$\Rightarrow \quad e = 0 \quad \& \quad a = -c(p \cdot n)$$

Similarly, Imposing $n_{\mu}d^{\mu\nu} = 0$ we get following result :

$$\Rightarrow \quad d = 0 \quad \& \quad a = -b(p \cdot n)$$

Now, $d^{\mu\nu}$ can be expressed as follows : (where $p_{\nu}d^{\mu\nu} = n_{\nu}d^{\mu\nu} = 0$ is also satisfied)

$$d^{\mu\nu}(p,n) = a \cdot g^{\mu\nu} - a \cdot \frac{(p^{\mu}n^{\nu} + p^{\nu}n^{\mu})}{p \cdot n}$$

Now we calculate following :

$$g_{\rho\nu}d^{\mu\nu} = d^{\mu}_{\rho} = a \cdot \delta^{\mu}_{\rho} - a \cdot \frac{(p^{\mu}n_{\rho} + p_{\rho}n^{\mu})}{p \cdot n}$$

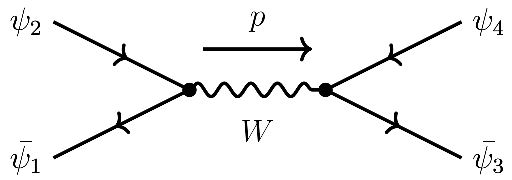
$$\Rightarrow \quad d^{\mu}_{\mu} = a \cdot \delta^{\mu}_{\mu} - a \cdot \frac{(p^{\mu}n_{\mu} + p_{\mu}n^{\mu})}{p \cdot n} = 4a - 2a = 2 \cdot a$$

$$\Rightarrow -2 = 2 \cdot a$$

Therefore we get $\Rightarrow \ a = -1$. Therefore we get out final expression as follows :

$$d^{\mu\nu}(p,n) = -g^{\mu\nu} + \frac{(p^{\mu}n^{\nu} + p^{\nu}n^{\mu})}{p \cdot n}$$

• Exercise 6.1 :



Above Diagram gives the Interaction between 4 fermions through W Boson Exchange. Using the general Feynman Rules, The Diagram can be evaluted as :

$$\mathcal{M} = \left(\bar{\psi}_1 \frac{ig}{\sqrt{2}} \cdot \frac{1-\gamma_5}{2}\gamma^\mu \psi_2\right) \frac{-g_{\mu\nu}}{p^2 - M_W^2} \left(\bar{\psi}_3 \frac{ig}{\sqrt{2}} \cdot \frac{1-\gamma_5}{2}\gamma^\nu \psi_4\right)$$

$$\Rightarrow \quad \mathcal{M} = \frac{g^2}{8} \cdot \frac{1}{p^2 - M_W^2} \cdot \left(\bar{\psi}_1 (1-\gamma_5)\gamma^\mu \psi_2\right) \cdot \left(\bar{\psi}_3 (1-\gamma_5)\gamma_\mu \psi_4\right)$$

Assuming $M_W^2 >> p^2$ and expanding as follows :

$$\Rightarrow \quad \frac{1}{p^2 - M_W^2} = \frac{-1}{M_W^2} \cdot \left(1 + (\frac{p^2}{M_W^2}) + (\frac{p^2}{M_W^2})^2 + \cdots\right)$$

Neglecting Higher order terms $\mathcal{O}(\frac{1}{M_W^4})$ we can write

$$\Rightarrow \quad \mathcal{M} = \frac{-g^2}{8} \cdot \frac{1}{M_W^2} \cdot \left(\bar{\psi}_1 (1-\gamma_5)\gamma^\mu \psi_2\right) \cdot \left(\bar{\psi}_3 (1-\gamma_5)\gamma_\mu \psi_4\right)$$

Therefore, In an 6-Dimensional EFT Expansion with $\Lambda = M_W$, The corresponding Wilson coefficient $c^{(6)}$ is given as :

$$\Rightarrow \quad c^{(6)} = \frac{-g^2}{8}$$

where g is the $SU(2)$ gauge cpupling.

• Exercise 6.2 :

Expression for $K_{QCD}(s)$ In terms of bare coupling constant α_s^0 is given as :

$$K_{QCD}(s) = 1 + \frac{\alpha_s^0}{\pi} + \left(\frac{\alpha_s^0}{\pi}\right)^2 \left[c + b_0\pi \log\left(\frac{\Lambda_{UV}^2}{s}\right)\right] + \mathcal{O}\big((\alpha_s^0)^3\big)$$

The renormalized coupling constant at renormalization scale μ is given as :

$$\alpha_s(\mu) = \alpha_s^0 + b_0 \log\left(\frac{\Lambda_{UV}^2}{\mu^2}\right)\alpha_s^2$$

We will be dropping all terms of order $\mathcal{O}(\alpha_s^3)$. Therefore, the α_s^2 in the expression can be taken as bare coupling constant or renormalized coupling constant.

Now, the expression in terms of Renormalized coupling can be given as follows :

$$\Rightarrow \quad K_{QCD}(\alpha_s(\mu), \mu^2/s) = 1 + \frac{1}{\pi}\cdot\left(\alpha_s(\mu) - b_0\log\left(\frac{\Lambda_{UV}^2}{\mu^2}\right)\alpha_s^2(\mu)\right) + \frac{1}{\pi^2}\cdot\left(\alpha_s(\mu) - b_0\log\left(\frac{\Lambda_{UV}^2}{\mu^2}\right)\alpha_s^2(\mu)\right)^2\left[c + b_0\pi\log\left(\frac{\Lambda_{UV}^2}{s}\right)\right] + \mathcal{O}\big(\alpha_s^3\big)$$

$$\Rightarrow \quad K_{QCD}(\alpha_s(\mu), \mu^2/s) = 1 + \left(\frac{\alpha_s(\mu)}{\pi}\right) + \frac{\alpha_s^2(\mu)}{\pi^2}\cdot\left[b_0\pi\log\left(\frac{\mu^2}{\Lambda_{UV}^2}\right)\right] + \frac{\alpha_s^2(\mu)}{\pi^2}\cdot\left[c + b_0\pi\log\left(\frac{\Lambda_{UV}^2}{s}\right)\right] + \mathcal{O}\big(\alpha_s^3\big)$$

$$\Rightarrow \quad K_{QCD}(\alpha_s(\mu), \mu^2/s) = 1 + \frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2\cdot\left[c + b_0\pi\log\left(\frac{\mu^2}{s}\right)\right] + \mathcal{O}\big(\alpha_s^3\big)$$

Is the Required expression in terms of the re-normalized coupling.

• Exercise 6.3 :

Higgs potential given as :

$$V(H) = \lambda\Big(H^\dagger H - \frac{v^2}{2}\Big)^2 - C_H(H^\dagger H)^3$$

Differentiating w.r.to $H^\dagger H$ and setting it to zero we get :

$$2\lambda\Big(H^\dagger H - \frac{v^2}{2}\Big) = 3C_H(H^\dagger H)^2$$

Solving the Quadratic we get :

$$H^\dagger H = \frac{\lambda}{3C_H}\Big(1 \pm \sqrt{1 - \frac{3C_H v^2}{\lambda}}\Big)$$

Using the second derivative, It can be determined that for $H^\dagger H < \frac{\lambda}{3C_H}$, The potential will have minima. Therefore, Potential will have a minimum at following point :

$$H^\dagger H = \frac{\lambda}{3C_H}\Big(1 - \sqrt{1 - \frac{3C_H v^2}{\lambda}}\Big)$$

Approximated to first order in C_H we get :

$$\Rightarrow \quad H^\dagger H = \frac{v^2}{2} + \frac{3C_H v^4}{8\lambda} = v^2 \cdot \Big(\frac{v_T}{v} - \frac{1}{2}\Big)$$

where,

$$v_T = v \cdot \Big(1 + \frac{3C_H v^2}{8\lambda}\Big)$$

• Exercise 7.1 :

We know the Naive Degree of Divergence (ω) for a Integral is given by :

$$\omega = DL - 2P + 2 \cdot \left(\sum_l \lfloor n_l/2 \rfloor \right)$$

where $\lfloor n_l/2 \rfloor$ is nearest integer less than or equal to $n_l/2$. n_l is the number of factors of loop momentun appearing in the Numerator. L is the number of Loops and P is the number of Propagators. (We are working in D Dimensions)

For Question 1, Tadpole Integral with mass : $L = 1, P = 1$ and $n_l = 0$.

$$\Rightarrow \quad \omega = D - 2$$

Therefore the Integral is UV Divergent. (If we are working in $D = 4$ or Higher Dimensions.)

For Question 2, Massive Bubble Integral : $L = 1, P = 2$ and $n_l = 2$.

$$\Rightarrow \quad \omega = D - 2$$

Therefore the Integral is UV Divergent. (If we are working in $D = 4$ or Higher Dimensions.)

For Question 3(a), Triangle Integral : $L = 1, P = 3$ and $n_l = 0$.

$$\Rightarrow \quad \omega = D - 6$$

Therefore the Integral isn't UV Divergent. Unless we are working in Higher Dimensions. ($D \geq 6$)

For Question 3(b), Triangle Integral : $L = 1, P = 3$ and $n_l = 2$.

$$\Rightarrow \quad \omega = D - 4$$

Therefore the Integral is UV Divergent. (If we are working in $D = 4$ or Higher Dimensions.)

For Question 4(a), Box Integral : $L = 1, P = 4$ and $n_l = 0$.

$$\Rightarrow \quad \omega = D - 8$$

Therefore the Integral isn't UV Divergent. Unless we are working in Higher Dimensions. ($D \geq 8$)

For Question 4(b), Box Integral : $L = 1, P = 4$ and $n_l = 2$.

$$\Rightarrow \quad \omega = D - 6$$

Therefore the Integral isn't UV Divergent. Unless we are working in Higher Dimensions. ($D \geq 6$)

For Question 3(c), Box Integral : $L = 1, P = 4$ and $n_l = 4$.

$$\Rightarrow \quad \omega = D - 4$$

Therefore the Integral is UV Divergent. (If we are working in $D = 4$ or Higher Dimensions.)

• Exercise 7.2 :

We have the Tensor $A_{\mu_1\cdots\mu_6}$ given as :

$$A_{\mu_1\cdots\mu_6} = Tr(\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_5}\gamma_5\gamma_{\mu_6}) + Tr(\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_5}\gamma_{\mu_6}\gamma_5)$$

We will be using the Identities $Tr(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5) = 4i\epsilon_{\mu\nu\rho\sigma}$ and $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. Using these recursively, the first term can be expressed as follows : (Also use cylicity of Trace)

$$Tr(\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_5\gamma_{\mu_6}) = Tr(\gamma_{\mu_6}\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_5)$$
$$\Rightarrow \quad = 2g_{\mu_6\mu_1}\cdot Tr(\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_5) - Tr(\gamma_{\mu_1}\gamma_{\mu_6}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_5)$$
$$\Rightarrow \quad = 2g_{\mu_6\mu_1}\cdot 4i\epsilon_{\mu_2\mu_3\mu_4\mu_5} - Tr(\gamma_{\mu_1}\gamma_{\mu_6}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_5)$$

Doing this recursively,

$$\Rightarrow \quad = 2g_{\mu_6\mu_1}\cdot 4i\epsilon_{\mu_2\mu_3\mu_4\mu_5} - 2g_{\mu_6\mu_2}\cdot 4i\epsilon_{\mu_1\mu_3\mu_4\mu_5} + Tr(\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_6}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_5)$$

$$\Rightarrow \quad = 2g_{\mu_6\mu_1}\cdot 4i\epsilon_{\mu_2\mu_3\mu_4\mu_5} - 2g_{\mu_6\mu_2}\cdot 4i\epsilon_{\mu_1\mu_3\mu_4\mu_5} + 2g_{\mu_6\mu_3}\cdot 4i\epsilon_{\mu_1\mu_2\mu_4\mu_5} - Tr(\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_6}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_5)$$

$$\Rightarrow \quad = 2g_{\mu_6\mu_1}\cdot 4i\epsilon_{\mu_2\mu_3\mu_4\mu_5} - 2g_{\mu_6\mu_2}\cdot 4i\epsilon_{\mu_1\mu_3\mu_4\mu_5} + 2g_{\mu_6\mu_3}\cdot 4i\epsilon_{\mu_1\mu_2\mu_4\mu_5}$$
$$- 2g_{\mu_6\mu_4}\cdot 4i\epsilon_{\mu_1\mu_2\mu_3\mu_5} + Tr(\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_6}\gamma_{\mu_5}\gamma_5)$$

$$\Rightarrow \quad = 2g_{\mu_6\mu_1}\cdot 4i\epsilon_{\mu_2\mu_3\mu_4\mu_5} - 2g_{\mu_6\mu_2}\cdot 4i\epsilon_{\mu_1\mu_3\mu_4\mu_5} + 2g_{\mu_6\mu_3}\cdot 4i\epsilon_{\mu_1\mu_2\mu_4\mu_5}$$
$$- 2g_{\mu_6\mu_4}\cdot 4i\epsilon_{\mu_1\mu_2\mu_3\mu_5} + 2g_{\mu_6\mu_5}\cdot 4i\epsilon_{\mu_1\mu_2\mu_3\mu_4} - Tr(\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_{\mu_6}\gamma_5)$$

Therefore we can observe that :

$$\frac{1}{8i}\cdot A_{\mu_1\cdots\mu_6} = g_{\mu_6\mu_1}\cdot\epsilon_{\mu_2\mu_3\mu_4\mu_5} - g_{\mu_6\mu_2}\cdot\epsilon_{\mu_1\mu_3\mu_4\mu_5} + g_{\mu_6\mu_3}\cdot\epsilon_{\mu_1\mu_2\mu_4\mu_5} - g_{\mu_6\mu_4}\cdot\epsilon_{\mu_1\mu_2\mu_3\mu_5} + g_{\mu_6\mu_5}\cdot\epsilon_{\mu_1\mu_2\mu_3\mu_4}$$

Contracting both sides with $g^{\mu_1\mu_6}$ we get : (Use : $g_{\rho\nu}g^{\mu\nu} = \delta^\mu_\rho$ & $\delta^\nu_\mu\epsilon_{\nu\rho\sigma\tau} = \epsilon_{\mu\rho\sigma\tau}$ & $\delta^\mu_\mu = D$)

$$\frac{1}{8i}\cdot A_{\mu_1\cdots\mu_6}g^{\mu_1\mu_6} = D\cdot\epsilon_{\mu_2\mu_3\mu_4\mu_5} - \epsilon_{\mu_2\mu_3\mu_4\mu_5} + \epsilon_{\mu_3\mu_2\mu_4\mu_5} - \epsilon_{\mu_4\mu_2\mu_3\mu_5} + \epsilon_{\mu_5\mu_2\mu_3\mu_4}$$

Using Anti-Symmetricity of Levi-Civita symbol we get :

$$\frac{1}{8i}\cdot A_{\mu_1\cdots\mu_6}g^{\mu_1\mu_6} = (D-4)\cdot\epsilon_{\mu_2\mu_3\mu_4\mu_5}$$

Now, If $\{\gamma_\mu, \gamma_5\} = 0$ was true, Our Tensor $A_{\mu_1\cdots\mu_6}$ must be zero. Therefore, above expression must also be zero. However, above expression can be arbitrarily zero, only if $D = 4$. (For arbitrary Tensor components $\mu_2\cdots\mu_5$, the levi-civita tensor cant be zero). Therefore, outside of 4 Dimensions, The Anti-commutation property of γ_5 cannot be realized without giving up cyclicity of the Trace property.

• Exercise 8.1 :

We know the Schwinger Parametrization is Given by :

$$\frac{1}{\Pi_{i=1}^{N} d_i^{n_i}} = \frac{1}{\Pi_{i=1}^{N}\Gamma(n_i)} \cdot \int_0^\infty \Big(\prod_{j=1}^{N} d\alpha_j \alpha_j^{n_j-1}\Big) \cdot \exp\Big(-\sum_{i=1}^{N}\alpha_i d_i\Big)$$

If We define $z = \sum_{i=1}^{N}\alpha_i$, we can express the Integral in terms of a new set of Parameters x_j which can be defined as $x_j = \alpha_j/z$.
The Product of Differentials In terms of these Parameters can be expressed as :

$$\prod_{i=1}^{N} d\alpha_i = z^{N-1}dz\prod_{j=1}^{N} dx_j \cdot \delta\Big(1-\sum_{i=1}^{N} x_i\Big)$$

Substituting everything In the Integral we get :

$$\frac{1}{\Pi_{i=1}^{N} d_i^{n_i}} = \frac{1}{\Pi_{i=1}^{N}\Gamma(n_i)} \cdot \int_0^\infty \Big(\int_0^\infty z^{M-1}dz\Big) \cdot \Big(\prod_{j=1}^{N} dx_j x_j^{n_j-1}\Big) \cdot \delta\Big(1-\sum_{i=1}^{N} x_i\Big) \cdot \exp\Big(-z\sum_{i=1}^{N} x_i d_i\Big)$$

Where we have defined $M = \sum_{i=1}^{N} x_i$. Now we will use the following Gamma Function Property :

$$\int_0^\infty dz \cdot z^{M-1}\exp(-z\cdot b) = \frac{\Gamma(M)}{b^M}$$

To Finally Arrive at our expression for Feynman Parametrization :

$$\frac{1}{\Pi_{i=1}^{N} d_i^{n_i}} = \frac{\Gamma(\sum_{i=1}^{N} x_i)}{\Pi_{i=1}^{N}\Gamma(n_i)} \cdot \int_0^\infty \Big(\prod_{j=1}^{N} dx_j x_j^{n_j-1}\Big) \cdot \delta\Big(1-\sum_{i=1}^{N} x_i\Big) \cdot \frac{1}{[\sum_{i=1}^{N} x_i d_i]^{\sum_{i=1}^{N} x_i}}$$

• Exercise 8.2 :

To Evaluate the Virtual QCD Corrections in $e^+e^- \to q\bar{q}$ Process, we first need to calculate the following 1-Loop 3-Point Integral :

$$\frac{(2\pi)^D}{i\pi^{D/2}} \cdot I_3 = 8(1-\epsilon) \cdot \int \frac{d^Dk}{i\pi^{D/2}} \cdot \frac{[Q^4 - 4(k\cdot q)(k\cdot\bar{q}) - 2k\cdot(q-\bar{q})Q^2 + \epsilon k^2 Q^2]}{[k^2]\cdot[(k+q)^2]\cdot[(k-\bar{q})^2]}$$

Where we have massless quarks such that $Q^2 = (q+\bar{q})^2 = 2q\cdot\bar{q}$.
Using The 3-Point Feynman Parametrization we can write :

$$\frac{(2\pi)^D}{i\pi^{D/2}} \cdot I_3 = 8(1-\epsilon)\Gamma(3) \cdot \int \frac{d^Dk}{i\pi^{D/2}} \int d\mathcal{Z} \cdot \frac{[Q^4 - 4(k\cdot q)(k\cdot\bar{q}) - 2k\cdot(q-\bar{q})Q^2 + \epsilon k^2 Q^2]}{[k^2 + 2z_2 k\cdot q - 2z_3 k\cdot\bar{q}]^3}$$

where we have defined :

$$\int d\mathcal{Z} = \int_0^\infty dz_1 dz_2 dz_3 \cdot \delta(1 - z_1 - z_2 - z_3)$$

. We shift the loop momentum such that we have a Perfect Quadratic In the Denominator. Therefore Substitute $l = k + z_2 q - z_3\bar{q}$. The Denominator will take form $[l^2 - R^2]^3$ where $R^2 = -z_2 z_3 Q^2$. Now we will also have to express the Numerator In terms of the l loop momentum. After some Algebra, We can establish following relations :

$$k = l - z_2 q + z_3 \bar{q}$$

$$2k\cdot q = 2l\cdot q + z_3 Q^2$$

$$2k\cdot\bar{q} = 2l\cdot\bar{q} - z_2 Q^2$$

$$k^2 = l^2 - 2z_2 l\cdot q + 2z_3 l\cdot\bar{q} - z_2 z_3 Q^2$$

Using the above substitutions, We can write The Numerator as follows :

$$A = [Q^4 - 4(k\cdot q)(k\cdot\bar{q}) - 2k\cdot(q-\bar{q})Q^2 + \epsilon k^2 Q^2] = Q^4\Big(1 + z_2 z_3(1-\epsilon) - z_2 - z_3\Big)$$

$$+Q^2\Big(\epsilon l^2 - 2l\cdot q(1 - z_2(1-\epsilon)) + 2l\cdot\bar{q}(1 - z_3(1-\epsilon))\Big) - 4(l\cdot q)(l\cdot\bar{q})$$

Our Integral Will now be Expressed as :

$$\frac{(2\pi)^D}{i\pi^{D/2}} \cdot I_3 = 8(1-\epsilon)\Gamma(3) \cdot \int d\mathcal{Z} \int \frac{d^Dl}{i\pi^{D/2}} \cdot \frac{A}{[l^2 + z_2 z_3 Q^2]^3}$$

Now we will use Following set of Identities to simplify the expression :

$$\int \frac{d^Dl}{i\pi^{D/2}} \cdot \frac{(l^2)^r}{[l^2 - R^2]^N} = (-1)^{N+r} \frac{\Gamma(r+\frac{D}{2})\Gamma(N-r-\frac{D}{2})}{\Gamma(\frac{D}{2})\Gamma(N)} \cdot (R^2)^{r-N+\frac{D}{2}}$$

$$\int \frac{d^Dl}{i\pi^{D/2}} \cdot \frac{l^\mu}{[l^2 - R^2]^N} = 0 \quad \& \quad \int \frac{d^Dl}{i\pi^{D/2}} \cdot \frac{l^\mu l^\nu}{[l^2 - R^2]^N} = \frac{g^{\mu\nu}}{D} \int \frac{d^Dl}{i\pi^{D/2}} \cdot \frac{l^2}{[l^2 - R^2]^N}$$

Now if we define Two Loop Integrals as follows :

$$L_1 = \int \frac{d^Dl}{i\pi^{D/2}} \cdot \frac{1}{[l^2 + z_2 z_3 Q^2]^3} = (-1) \cdot \frac{\Gamma(1+\epsilon)}{\Gamma(3)} \cdot (-Q^2)^{-1-\epsilon} \cdot (z_2 z_3)^{-1-\epsilon}$$

$$L_2 = \int \frac{d^Dl}{i\pi^{D/2}} \cdot \frac{l^2}{[l^2 + z_2 z_3 Q^2]^3} = \frac{(2-\epsilon)\Gamma(\epsilon)}{\Gamma(3)} \cdot (-Q^2)^{-\epsilon} \cdot (z_2 z_3)^{-\epsilon}$$

We can write the Following Integral as :

$$\int \frac{d^D l}{i\pi^{D/2}} \cdot \frac{-4(l\cdot q)(l\cdot \bar{q})}{[l^2 + z_2 z_3 Q^2]^3} = \frac{-2Q^2}{D} \cdot L_2 = \frac{Q^2}{(\epsilon - 2)} \cdot L_2$$

Now we can Write our Original Integrals as follows :

$$\frac{(2\pi)^D}{i\pi^{D/2}} \cdot I_3 = 8(1-\epsilon)\Gamma(3) \cdot \int d\mathcal{Z} \Big[Q^4 \Big(1 + z_2 z_3(1-\epsilon) - z_2 - z_3\Big) \cdot L_1 + \epsilon Q^2 \cdot L_2 + \frac{Q^2}{\epsilon - 2} \cdot L_2 \Big]$$

$$\frac{(2\pi)^D}{i\pi^{D/2}} \cdot I_3 = 8(1-\epsilon)\Gamma(3) \cdot \int d\mathcal{Z} \Big[Q^4 \Big(1 + z_2 z_3(1-\epsilon) - z_2 - z_3\Big) \cdot L_1 + \frac{(1-\epsilon)^2 Q^2}{\epsilon - 2} \cdot L_2 \Big]$$

Substituting the L_1 and L_2 expressions, and simplifying we get :

$$\frac{(2\pi)^D}{i\pi^{D/2}} \cdot I_3 = 8(1-\epsilon)\frac{Q^2\ \Gamma(\epsilon)}{(-Q^2)^\epsilon} \cdot \int d\mathcal{Z} \cdot f(z_2, z_3)$$

where we have defined :

$$f(z_2, z_3) = \epsilon \cdot \Big(1 + z_2 z_3(1-\epsilon) - z_2 - z_3\Big)(z_2 z_3)^{-1-\epsilon} - (1-\epsilon)^2 (z_2 z_3)^{-\epsilon}$$

$$f(z_2, z_3) = \epsilon \cdot (z_2 z_3)^{-1-\epsilon} - \epsilon (z_3)^{-1-\epsilon}(z_2)^{-\epsilon} - \epsilon (z_3)^{-\epsilon}(z_2)^{-1-\epsilon} - (1-2\epsilon)(1-\epsilon)(z_2 z_3)^{-\epsilon}$$

Let us calculate the $\mathcal{Z}$ Integral Explicitely. We will first Integrate over z_1 because of which delta function will put a constraint $0 < (z_2 + z_3) < 1$. Therefore we have a 2 dimensional Integral over the area bounded by $z_2 = 0, z_3 = 0$ & $z_2 + z_3 = 1$. As a result, we can calculate it as follows :

$$\int d\mathcal{Z} f(z_2, z_3) = \int_0^1 dz_3 \int_0^{1-z_3} dz_2 \cdot f(z_2, z_3)$$

After evaluating the Integral over z_2 with simple power rule, The Integrals with respect to z_3 can be written in Terms of Euler-Beta Function. Our $\mathcal{Z}$ Integral then can be written as sum of 4 Beta functions as follows :

$$\int d\mathcal{Z} f(z_2, z_3) = -B(-\epsilon, 1-\epsilon) - \frac{\epsilon}{(1-\epsilon)} \cdot B(-\epsilon, 2-\epsilon) + B(1-\epsilon, 1-\epsilon) - (1-2\epsilon)\cdot B(1-\epsilon, 2-\epsilon)$$

Finally, after simplifying and some algebraic manipulation using Gamma Function properties, we can express our $\mathcal{Z}$ Integral as follows :

$$\int d\mathcal{Z} f(z_2, z_3) = \Big(\epsilon + \frac{1}{\epsilon} - \frac{1}{2}\Big) \cdot \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}$$

Substituting in our Original Integral we get :

$$\frac{(2\pi)^D}{i\pi^{D/2}} \cdot I_3 = 8(1-\epsilon)\frac{Q^2\ \Gamma(\epsilon)}{(-Q^2)^\epsilon} \cdot \Big(\epsilon + \frac{1}{\epsilon} - \frac{1}{2}\Big) \cdot \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}$$

Again Simplifying with Gamma Function properites : $\Gamma(1+\epsilon) = \epsilon \cdot \Gamma(\epsilon)$,
We can write our result as follows :

$$\frac{(2\pi)^D}{i\pi^{D/2}} \cdot I_3 = \frac{4Q^2}{(-Q^2)^\epsilon} \cdot \Big[\ \frac{2}{\epsilon^2} - \frac{3}{\epsilon} + 3 - 2\epsilon\ \Big] \cdot \frac{\Gamma^2(1-\epsilon)\cdot\Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)}$$

• Exercise 9.1 :

We have our Propagator Momenta defined as : (where $p_1 + p_2 + p_3 = 0$)

$$q_1 = k + p_1 = q_3 + p_1 \quad q_2 = q_1 + p_2 \quad q_3 = k$$

Therefore, The General Kinematic Matrix can be given by :

$$S_{ij} = (q_i - q_j)^2 - m_i^2 - m_j^2$$

$$S_{ij} = \begin{pmatrix} -2m_1^2 & p_2^2 - m_1^2 - m_2^2 & p_1^2 - m_1^2 - m_3^2 \\ p_2^2 - m_1^2 - m_2^2 & -2m_2^2 & p_3^2 - m_2^2 - m_3^2 \\ p_1^2 - m_1^2 - m_3^2 & p_3^2 - m_2^2 - m_3^2 & -2m_3^2 \end{pmatrix}$$

Now, The Condition for soft singularity is given as : (for at least one value of i)

$$S_{i+1i+1} = S_{i+1i+2} = S_{i+1i} = 0$$

For $i = 1, 2, 3$ we get :

$$S_{22} = S_{23} = S_{21} = 0 \quad OR \quad S_{33} = S_{34} = S_{32} = 0 \quad OR \quad S_{44} = S_{45} = S_{43} = 0$$

Now, The Condition for collinear singularity is given as : (for at least one value of i)

$$S_{ii} = S_{i+1i+1} = S_{i+1i} = 0 \quad (modN)$$

For $i = 1, 2, 3$ we get :

$$S_{11} = S_{22} = S_{21} = 0 \quad OR \quad S_{22} = S_{33} = S_{32} = 0 \quad OR \quad S_{33} = S_{11} = S_{13} = 0$$

Now let us Investigate all the cases :

Case 1 :

$$S_{ij} = \begin{pmatrix} 0 & p_2^2 & p_1^2 \\ p_2^2 & 0 & p_3^2 \\ p_1^2 & p_3^2 & 0 \end{pmatrix}$$

Therefore, we don't have any of the Divergences.

Case 2 :

$$S_{ij} = \begin{pmatrix} -2m^2 & p_2^2 - m^2 & p_1^2 - m^2 \\ p_2^2 - m^2 & 0 & p_3^2 \\ p_1^2 - m^2 & p_3^2 & 0 \end{pmatrix}$$

Therefore, we don't have any of the Divergences.

Case 3 :

$$S_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_3^2 \\ 0 & p_3^2 & 0 \end{pmatrix}$$

Therefore, we have the Collinear Divergence, because condition satisfied for two of the indices.

Case 4 :

$$S_{ij} = \begin{pmatrix} 0 & p_2^2 & 0 \\ p_2^2 & 0 & p_3^2 \\ 0 & p_3^2 & 0 \end{pmatrix}$$

Therefore, we have the Collinear Divergence, because condition satisfied for one of the indices.

Case 5 :

$$S_{ij} = \begin{pmatrix} 0 & p_2^2 - m^2 & 0 \\ p_2^2 - m^2 & -2m^2 & p_3^2 - m^2 \\ 0 & p_3^2 - m^2 & 0 \end{pmatrix}$$

Therefore, we have the Collinear Divergence, because condition satisfied for one of the indices.

Case 6 :

$$S_{ij} = \begin{pmatrix} 0 & p_2^2 - m^2 & 0 \\ p_2^2 - m^2 & -2m^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, we have the Collinear Divergence, because condition satisfied for one of the indices. We also have Soft divergence because condition satisfied for one of the indices.

Case 7 :

$$S_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2m^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, we have the Collinear Divergence, because condition satisfied for one of the indices. We also have Soft divergence because condition satisfied for one of the indices.

Case 8 :

$$S_{ij} = \begin{pmatrix} -2m_1^2 & -m_1^2 & 0 \\ -m_1^2 & -2m^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, We have Soft divergence because condition satisfied for one of the indices.

Summarizing,
We have Soft Divergence for the Cases : 6,7,8 And Collinear Divergence for the cases : 3,4,5,6,7.

• Exercise 11.1 :

We consider Deep Inelastic Scattering (DIS) for the process : $e(k) + q(p) \to e(k') + q'(p')$ such that the electrons and quarks are massless. The standard varible definitions we will be using are :

$$s = (P+k)^2, \quad q^\mu = k^\mu - k'^\mu, \quad Q^2 = -q^2 = 2MExy, \quad x = \frac{Q^2}{2P\cdot q},$$

$$y = \frac{P\cdot q}{P\cdot k} = 1 - \frac{E'}{E}, \quad v = \frac{P\cdot q}{M} = E - E', \quad \hat{s} = (p+k)^2, \quad p^\mu = \xi P^\mu, \quad Q^2 = y\cdot\hat{s}$$

Now, we know the general Phase Space for 2 final state particles can be given by :

$$d\Phi_2 = \frac{1}{(2\pi)^2}\cdot d^4k'\delta(k'^2)\Theta(E')\cdot d^4p'\delta(p'^2)\Theta(E_{p'})\cdot\delta(k'+p'-k-p)$$

Now, we Integrate over p' using the last delta function, imposing $p' = q + p \Rightarrow p'^2 = 2p\cdot q - Q^2$. We also write $d^4k'\delta(k'^2)\Theta(E') = \frac{d^3k'}{2E'}$. Therefore, we get following :

$$d\Phi_2 = \frac{1}{(2\pi)^2}\cdot\frac{d^3k'}{2E'}\cdot\delta(2p\cdot q - Q^2)$$

Now, we use the $q = k - k'$ relation to find the expression for $\cos\theta$:

$$q^2 = -2k\cdot k' = -2EE'(1-\cos\theta)$$

$$\Rightarrow \quad \cos\theta = 1 - \frac{Q^2}{2EE'} = 1 - \frac{xyM}{(1-y)E}$$

Now we can use the following relation :

$$\frac{d^3k'}{2E'} = (2\pi)^3\cdot\frac{d\phi}{2\pi}\cdot\frac{E'}{8\pi^2}\cdot dE'\cdot d\cos\theta = (2\pi)^3\cdot\frac{d\phi}{2\pi}\cdot\frac{EMy}{8\pi^2}\cdot dy\cdot dx$$

along with

$$\delta(2p\cdot q - Q^2) = \frac{x}{Q^2}\delta(\xi - x)$$

Substituting everything back into the original equation, we get our result :

$$d\Phi_2 = \frac{d\phi}{(4\pi)^2}\cdot dy\cdot dx\cdot\delta(\xi - x)$$

• Exercise 11.2 :

We consider the process $\gamma^* q \to q' g$ with massless quarks. The Matrix Element is given as :

$$\sum |\mathcal{M}(\gamma^* q \to q' g)|^2 = e_q^2 \alpha_s \mu^{2\epsilon} C_F N_c 8(1-\epsilon) \cdot \Big\{ (1-\epsilon)\Big[\frac{g\cdot q}{g\cdot q'} + \frac{g\cdot q'}{g\cdot q}\Big] + \frac{Q^2(q\cdot q')}{(g\cdot q)(g\cdot q')} + 2\epsilon \Big\}$$

We use parametrizations as follows :

$$2g\cdot q' = \frac{Q^2}{z}(1-z), \quad 2g\cdot q = \frac{Q^2}{z}v, \quad 2q\cdot q' = \frac{Q^2}{z}(1-v),$$

$$v = \frac{(1+\cos\theta)}{2}, \quad \hat{s} = (g+q')^2 = \frac{Q^2}{z}(1-z)$$

Substituting these relations back, we get the following result :

$$\sum |\mathcal{M}(\gamma^* q \to q' g)|^2 = e_q^2 \alpha_s \mu^{2\epsilon} C_F N_c 8(1-\epsilon) \cdot \Big\{ (1-\epsilon)\Big[\frac{v}{(1-z)} + \frac{(1-z)}{v}\Big] + \frac{2z}{(1-z)} \cdot \frac{(1-v)}{v} + 2\epsilon \Big\}$$

Now, we know, cross-section can be given as follows :

$$d\sigma = \frac{1}{flux} \cdot \frac{1}{4} \cdot \sum |\mathcal{M}(\gamma^* q \to q' g)|^2 \cdot d\Phi_2$$

where, The Phase space Integral is given by :

$$\int d\Phi_2 = \frac{1}{(8\pi)} \cdot \Big(\frac{4\pi}{\hat{s}}\Big)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \int_0^1 dv [v(1-v)]^{-\epsilon}$$

and the flux can be given as :

$$flux = 4(p_i \cdot p_j) = 4(q' + g - q)\cdot q = 2(2q\cdot q' + 2g\cdot q) = 2\frac{Q^2}{z}$$

Substituting Everything, we can write the expression for cross-section as follows :

$$\sigma = \frac{z}{8Q^2}\frac{1}{(8\pi)} \cdot \Big(\frac{4\pi}{\hat{s}}\Big)^{\epsilon} \cdot \frac{1}{\Gamma(1-\epsilon)} e_q^2 \alpha_s \mu^{2\epsilon} C_F N_c 8(1-\epsilon) \cdot f(z,\epsilon)$$

where, we will calcualte the $f(z,\epsilon)$ explicitely below :

$$f(z,\epsilon) = \frac{(1-\epsilon)}{(1-z)} \int_0^1 dv (v)^{1-\epsilon}(1-v)^{-\epsilon} + (1-\epsilon)(1-z)\int_0^1 dv (v)^{-\epsilon-1}(1-v)^{-\epsilon} + \frac{2z}{(1-z)}\int_0^1 dv (v)^{-\epsilon-1}(1-v)^{1-\epsilon} + 2\epsilon \int_0^1 dv (v)^{-\epsilon}(1-v)^{-\epsilon}$$

Expressing all the integrals as Beta Function, using Gamma function properties, with some algebra, we write $f(z,\epsilon)$ as follows :

$$f(z,\epsilon) = \frac{\Gamma^2(1-\epsilon)}{(1-2\epsilon)\Gamma(1-2\epsilon)} \cdot \Big[\frac{-1}{\epsilon} \cdot \Big(\frac{2z}{(1-z)} + (1-z)\Big) + \Big(\frac{1}{2(1-z)} + \frac{2z}{(1-z)} + 3(1-z)\Big) + \mathcal{O}(\epsilon)\Big]$$

We can write, (Absorbing the factor $(1-\epsilon)$ from earlier)

$$\frac{1}{(1-2\epsilon)} = 1 + 2\epsilon + \mathcal{O}(\epsilon^2), \quad (1-\epsilon)\cdot(1+2\epsilon) = (1+\epsilon) + \mathcal{O}(\epsilon^2)$$

Now, $f(z,\epsilon)$ can be written in a simplified form as follows :

$$f(z,\epsilon) = \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \cdot \left[\frac{-1}{\epsilon} \cdot \left(\frac{1+z^2}{1-z}\right) + \left(\frac{1}{2(1-z)} + 2(1-z)\right) + \mathcal{O}(\epsilon)\right]$$

Our Final Expression for cross-section can be written as :

$$\sigma = \frac{z}{8Q^2}\frac{1}{(8\pi)} \cdot \left(\frac{4\pi}{\hat{s}}\right)^{\epsilon} \cdot e_q^2 \alpha_s \mu^{2\epsilon} C_F N_c 8 \cdot \left[\frac{-1}{\epsilon} \cdot \left(\frac{1+z^2}{1-z}\right) + \left(\frac{1}{2(1-z)} + 2(1-z)\right) + \mathcal{O}(\epsilon)\right] \cdot \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

We can observe, that, Coefficient of $1/\epsilon$ is indeed proportional to the splitting function $P_{qq}(z)$.

• Exercise 12.1 :

The regularized splitting function is given by :

$$P_{q\to qg}(x) = C_F\Big(\frac{1+x^2}{[1-x]_+} + K\delta(1-x)\Big)$$

We can use the following relation to calculate value of K

$$\int_0^1 dx P_{q\to qg}(x) = 0$$

which Implies the Quark number conservation.

$$\Rightarrow \quad \int_0^1 dx \frac{1+x^2}{[1-x]_+} = -\int_0^1 dx K\delta(1-x) = -K$$

But we know,

$$\Rightarrow \quad \int_0^1 dx \frac{f(x)}{[1-x]_+} = \int_0^1 dx \frac{f(x)-f(1)}{(1-x)}$$

Substituting for our case we get,

$$\Rightarrow \quad \int_0^1 dx \frac{(x^2-1)}{1-x} = -K$$

Simplifying we get,

$$\Rightarrow \quad K = \frac{3}{2}$$

• Exercise 12.2 :

1. Best measured Value for the Higgs Boson mass by ATLAS is 124.97 ± 0.24 GeV, which is combined from Run-1 & Run-2 data with $H \to \gamma\gamma$ and $H \to 4l$ decay channels.

Best measured Value for the Higgs Boson mass by CMS is 125.38 ± 0.16 GeV, which is combined from Run-1 & 2016 data with $H \to \gamma\gamma$ and $H \to ZZ \to 4l$ decay channels.

2. Higgs Boson Production through Gluon Fusion has the least Uncertainty in the Cross-section. If we are considering cross-sections normalized to Gluon Fusion Cross-section, $t\bar{t}H + tH$ channel has least uncertainty.

3. The $t\bar{t}H + tH$ production channel Has statistical Uncertainty less than the Systematic Uncertainties. Production through WH just also satisfies the condition.

4. Signal Strengths for $H \to Z\gamma$ and $H \to \mu\mu$ are furthest away from SM Predictions.

5. Range of k_λ in which trilinear Higgs Boson Coupling is measured by CMS at 95% confidence level : $-3.3 < k_\lambda < 8.5$

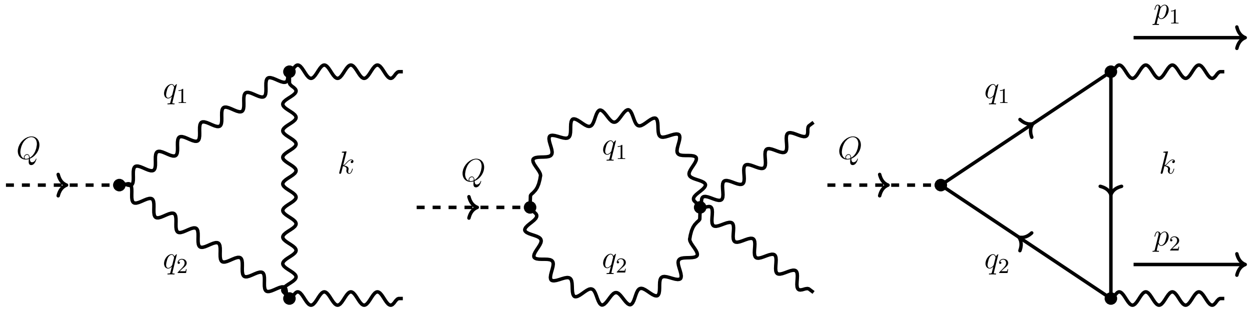
• Exercise 13.1 :

We Consider the Higgs to Diphoton Decay $H(Q) \to \gamma(p_1) + \gamma(p_2)$ in Rest Frame of Higgs :

$$Q = (m_H, 0, 0, 0)$$

$$p_1 = E(1, 0, 0, 1)$$

$$p_2 = E(1, 0, 0, -1)$$



We know the Decay Width will be given by :

$$d\Gamma = \frac{J}{2\sqrt{Q^2}} |\mathcal{M}|^2 d\Phi_2$$

For our case, $J = 1/2!$ because we have two photons in Final state. We can also simply evaluate the Phase space Integral as

$$\int d\Phi_2 = \frac{1}{8\pi}$$

Therefore,

$$\Rightarrow \quad \Gamma = \frac{1}{32 m_H} \cdot |\mathcal{M}|^2$$

Using the Feynman rules for the third diagram, we can write $\mathcal{M}$ as follows :

$$\mathcal{M} = \frac{i \cdot m_f}{v} \cdot \frac{e^2 \cdot e_f^2 \cdot N_{cf}}{(4\pi)^{D/2}} \cdot \epsilon_{1\mu}^* \epsilon_{2\nu}^* \int \frac{d^d k}{i\pi^{D/2}} \cdot \frac{Tr[\gamma^\mu (\not{q}_1 + m_f)(\not{q}_2 + m_f)\gamma^\nu (\not{k} + m_f)]}{[k^2 - m_f^2][q_1^2 - m_f^2][q_2^2 - m_f^2]}$$

$$\Rightarrow \quad \mathcal{M} = \epsilon_{1\mu}^* \epsilon_{2\nu}^* \cdot \mathcal{M}^{\mu\nu}$$

where, from the lecture notes, we have (We use same definition of $f(\tau)$ but with $\tau = (4m_f^2)/m_H^2$)

$$\mathcal{M}^{\mu\nu} = i\frac{\alpha}{4\pi} \frac{m_H^2}{v} \cdot N_{cf} e_f^2 \cdot \frac{A(\tau)}{2} \cdot T^{\mu\nu}$$

where, ($N_{cf} = 3$ for quarks and 1 for leptons)

$$A(\tau) = 2\tau[1 + (1-\tau)f(\tau)]$$

We consider all fermions, so we sum over f. Also, we multiply by factor of two accounting for the Identical Contibution from fermion-flow In opposite direction. Therefore, The Net Amplitude for Fermion contribution is :

$$\mathcal{M}^{\mu\nu} = i\frac{\alpha}{4\pi} \frac{m_H^2}{v} \cdot T^{\mu\nu} \cdot \Big(\sum_f N_{cf} e_f^2 A(\tau) \Big)$$

Because, the Other W Boson diagrams will also have the same Photon Polarisation Vectors with the $T^{\mu\nu}$, we can Take the Mod Squared of it and take that factor outside. (We Sum over Photon Polarisations which gives us factor of $g_{\mu\rho} g_{\nu\sigma}$)

$$\Rightarrow \quad |\epsilon^*_{1\mu}\epsilon^*_{2\nu} \cdot T^{\mu\nu}|^2 = g_{\mu\rho}g_{\nu\sigma}T^{\mu\nu}T^{\rho\sigma} = g_{\mu\rho}g_{\nu\sigma}(g^{\mu\nu} - \frac{p_2^\mu p_1^\nu}{p_1 \cdot p_2})(g^{\rho\sigma} - \frac{p_2^\rho p_1^\sigma}{p_1 \cdot p_2})$$

$$\Rightarrow \quad = (\delta^\nu_\rho - \frac{p_{2\rho}p_1^\nu}{p_1 \cdot p_2})(\delta^\rho_\nu - \frac{p_2^\rho p_{1\nu}}{p_1 \cdot p_2}) = D - 2 = 2 \quad (D \to 4)$$

We take this factor of two out, such that, Our Net Partial Decay Width can be given by above 3 Diagrams as follows :

$$\Gamma[H \to \gamma\gamma] = \frac{1}{16 m_H} \cdot |\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3|^2$$

Where we have already Evaluated the $\mathcal{M}_3$ as follows :

$$\Rightarrow \quad \mathcal{M}_3 = i\frac{\alpha}{4\pi}\frac{m_H^2}{v} \cdot \Big(\sum_f N_{cf} e_f^2 A(\tau)\Big)$$