Integration By Parts Identities and Scattering Amplitudes

Chaitanya Paranjape, IIT(ISM) Dhanbad

Abstract

This is a short report on the study of Integration By Parts Identities with emphasis on their mathematical structure and properties through various examples. Integration By Parts Identities are extensively used for reduction of the number of Scalar Feynman Integrals which need to be evaluated in the calculation of Scattering Amplitudes. This report is mostly based on the study of [1],[2],[3] and references therein. The aim of this report is to help novel researchers become familiar with the IBP reduction process fairly easily.

I am humbled and very much grateful to have had Dr. Taushif Ahmed's Guidance during my course of study. Invaluable Discussions with him helped me really understand the strucutre of IBP Identities and the research work involved in the area of scattering amplitude calculations.

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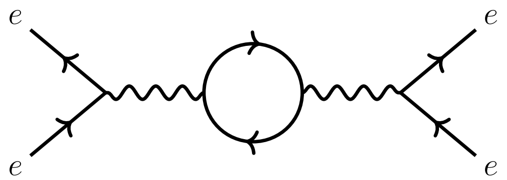
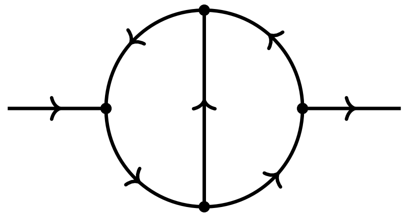
Feynman Diagrams

Scattering cross-sections are crucial to study elementary particle reactions at high energy colliders. They can be calculated theoretically with the framework of modern Quantum Field Theory and then put to test with the experimental results. In order to calculate these cross-sections, one has to calculate the scattering amplitude, which tells us about the Dynamics of the reaction. Instead of opting for rigorous QFT calculations, one can calculate these scattering amplitudes from corresponding Feynman Diagrams of Interaction. A typical Feynman diagram can have external legs, internal lines and loops.





Such a Feynman Diagram can be decomposed into 3 components each with different roles

- Information about polarisation of particles
- Pure tensorial or spinoral part
- A scalar integral factor enclosing internal line properties

This decomposition is possible for any diagram without loss of generality by the virtue of Lorentz in-variance and gauge in-variance of a physical Lagrangian.

The scalar Integral factors are commonly called as scalar Feynman Integrals. In order to calculate the scattering cross-section of a reaction, a large number of scalar Feynman integrals need to be evaluated. This number rapidly increases beyond thousands as soon as reaction becomes more and more complex. Evaluating all these integrals individually is not practically feasible. However, for a given topology of Feynman Integrals, there exists a set of Identities called as Integration By Parts Identities. These identities can establish relations between various Feynman Integrals and therefore can be used to express all the Integrals in terms of a much smaller subset called as Master Integrals. Master Integrals of a given topology are set of Feynman Integrals which can be used to express all the other scalar Feynman Integrals. Therefore, calculating a Scattering amplitude boils down to reduction of the scalar Feynman integrals in terms of Master integrals and evaluating these Master integrals (Master Integrals are also Scalar Feynman Integrals). Let us make ourselves familiar with the structure of a generic Scalar Feynman Integral.

Scalar Feynman Integrals

Every Scalar Feynman Integral has a corresponding Feynman Diagram. Let us introduce the terminology for a typical Scalar Feynman Integral.

Let the Diagram have (E+1) number of external legs, then the number of independent external momenta is equal to E. Set of these external momenta is given by :

$$\boldsymbol{P} = \{p_1, p_2, \ldots, p_E\ \}$$

If the Diagram has L loops, then the number of loop momenta is L and their set is given by :

$$\boldsymbol{K} = \{k_1, k_2, \ldots, k_L\ \ \}$$

The Total $L + E$ number of Momenta can be labelled as $q_i = k_1, \ldots, k_L, p_1, \ldots, p_E$
The Diagram can have I number of internal lines with momenta $l_1, \ldots, l_I$
which are just linear combinations of q_i

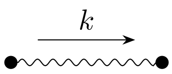
Now, we can define the Scalar Products of these Momenta as : $s_{ij} = q_i \cdot q_j \quad (j \geq i)$

For $i > L$, we get External Kinematic quantities. The Feynman Integral will be dependent on these quantities after the Integral evaluation.

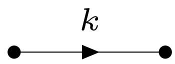
For $1 \leq i \leq L$, we will get Scalar Products corresponding to the Integration variables. These are what we will be dealing with along with the Propagators. The number of such Scalar products N is given by :

$$N = L \cdot E + \frac{L \cdot (L+1)}{2}$$

Propagators in the Diagram correspond to the Denominators of the Integral. For Example :



$$= \frac{1}{k^2 + i0}$$



$$= \frac{1}{k^2 + m^2 + i0}$$

Propagators can either be Massive or Massless. The Corresponding denominators have been demonstrated above. For I internal lines, we can label the denominators as $D_1, \ldots, D_I$.

These D_j terms can be used to express a total I number of Scalar Products, thus help to reduce the Integral in terms of D_j terms only.
For a Diagram with (E+1) external legs and L number of loops, I is given as :
I = 3L +E - 2
If the total number of Scalar Products N is greater than I, in that case, it is not possible to express all s_{ij} as linear combinations of the denominators. Then the Number of Ir-reducible Scalar Products is given by :

$$N \text{ - } I = \frac{(L-1)(L+2E-4)}{2}$$

Vacuum Diagrams (E=0) have to be considered separately. In case of Vacuum Diagrams, the corresponding I and N - I are given by :

$$I = 3(L - 1) \qquad N \text{ - } I = \frac{(L-2)(L-3)}{2}$$

We want all Scalar Products s_{ij} to be expressible as linear functions of the Denominators D_j. Therefore, we add irreducible numerators $D_{I+1}, \ldots, D_N$ which are linear functions of s_{ij} to make a complete set of Denominators $D_1, \ldots, D_N$ which are linearly Independent. As a result, The Integral can be expressed solely in terms of the $D_1, \ldots, D_N$. The corresponding expressions are given as :

$$D_a = \sum_{i=1}^{L} \sum_{j=i}^{L+E} A_a^{ij} s_{ij} + m_a^2$$

$$s_{ij} = \sum_{a=1}^{N} A_{ij}^a (D_a - m_a^2)$$

Here, the matrix A_a^{ij} is the inverse of A_{ij}^a.

Now we can define the Scalar Feynman Integral as Follows : (In Dimensional regularization)

$$J(n_1, \ldots, n_N) = \frac{1}{(i\pi^{d/2})^L} \int d^d k_1 \ldots d^d k_L f(k_1, \ldots, k_L, p_1, \ldots, p_E)$$

where,

$$f(k_1, \ldots, k_L, p_1, \ldots, p_E) = \frac{1}{D_1^{n_1} \cdots D_N^{n_N}}$$

For irreducible numerators $n_a \leq 0$, where a $\in [I+1, N]$.

Key thing to note here is that, argument of J is a point in an N-dimensional Integer space. Therefore, Scalar Feynman Integrals of a given Topology are associated with a point in $\mathbb{Z}^N$.

If $n_a \geq 0$, then D_a corresponds to the denominator of a propagator and if $n_a < 0$, then it corresponds to an irreducible numerator. Depending on whether n_a is positive or non-positive, the N-dimensional integer space can be divided into various sectors. For a given Topology of Feynman Integral, different sectors of the $\mathbb{Z}^N$ space correspond to the various sub-topologies of the given Topology. These sub-topologies are nothing but the Topologies of all the Integrals(Diagrams) which can be obtained by 'shrinking' the internal lines in the given Topology of Feynman Integral. Shrinking the internal lines does not affect the momentum conservation. Now let us understand the Mathematical prescription of sectors in the $\mathbb{Z}^N$ space.

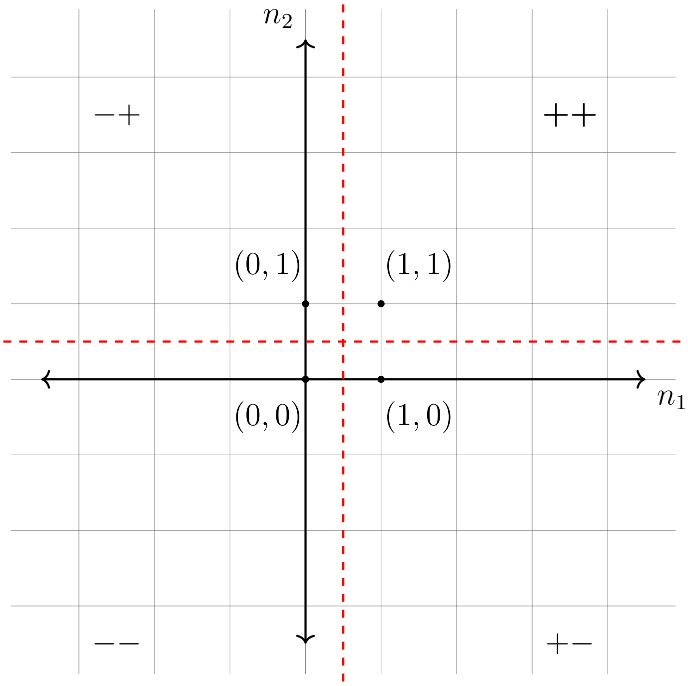
All the sectors are labelled as ($\theta_1, \ldots, \theta_N$). Where θ_i can either be 0 or 1. Therefore, number of sectors is given by 2^N. The sector ($\theta_1, \ldots, \theta_N$) is a set of all points ($n_1, \ldots, n_N$) in $\mathbb{Z}^N$ whose coordinates obey the condition :

$$sign(n_\alpha - 1/2) = 2\theta_\alpha - 1$$

Precisely, the point ($\theta_1, \ldots, \theta_N$) belongs to the ($\theta_1, \ldots, \theta_N$) Sector and is called as the corner point of the sector. We can see that, Integrals of the same sector have same number of denominators and thus powers of denominators will determine their positions in that sector.

It is Natural to consider integrals with less denominators to be simpler. Consequently, the pure negative sector (all $n_a \leq 0$) is the simplest because it is a trivial sector. If the Integral is zero at all points of a sector, i.e. $J = 0$ everywhere in a sector, then that sector is called as a trivial sector. The pure negative sector is always a trivial sector because a scaleless Integral is zero in dimensional regularization even with additional polynomial factors in numerator. Scaleless Integrals are those Integrals which gain an additional non-unity factor under some transformation.

Therefore, as we move to the sectors with higher number of denominators, complexity of the Integral increases. Some sectors can also be transformed into each other by symmetries.



In case of $N = 2$, We have $--, -+, +-, ++$ Sectors with $(0,0), (0,1), (1,0), (1,1)$ as the corresponding corner points. $--$ is a Trivial sector and thus the lowest in ordering. $++$ is the Highest sector which can be solved when its lower sectors $- + \, and + -$ have been solved. The sectors $- + \, and + -$ cannot be compared with each other for ordering. Ordering of the Integrals according to sectors plays a role in the IBP Reduction process. Speaking of IBPs, let us now Introduce the Integration By Parts Identities (IBP) of Scalar Feynman Integrals.

Integration By Parts

Integration By Parts Identities (IBPs) are a set of relations between Feynman Integrals of same Topology that are the virtue of dimensionally regularizaed Feynman Integrals. As the name suggests, they are reflection of the fact that the Integral of a total derivative is zero in dimensional regularization.

Let us consider the redefinition of loop momenta as follows :

$$k_i \rightarrow M_{ij}q_j = A_{ij}k_j + B_{ij}p_j$$

where, the Transformation Matrix M has the form :

$$M = \begin{pmatrix} A_{11} & \dots & A_{1L} & B_{11} & \dots & B_{1E} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{L1} & \dots & A_{LL} & B_{L1} & \dots & B_{LE} \end{pmatrix}$$

It is an $L \times (L+E)$ Transformation Matrix, provided that the substitution is invertible, i.e. det A $\neq 0$.

Now, we can consider the corresponding infinitesimal transformation and obtain the Feynman Integral in the new form. However, because Feynman Integrals stay invariant under such redefinition of loop momenta, we come across a set of Identities which are commonly called as the Integration By Parts Identities. As the Integration By Parts Identities correspond to the Infinitesimal Transformations, they also form Lie Algebra. Following are the expressions of IBPs and their operators :

$$O_{ij}J(n_1, \dots, n_N) = 0$$

$$\frac{1}{(i\pi^{d/2})^L} \int d^dk_1 \dots d^dk_L \; O_{ij}f \; = 0$$

where,

$$O_{ij} = \partial_i \cdot q_j = \frac{\partial}{\partial k_i} \cdot q_j = d\delta_{ij} + q_j \cdot \frac{\partial}{\partial k_i} \quad : \quad i \in [1, L], \quad j \in [1, L+E]$$

O_{ij} are set of operators which also constitute Lie Algebra [3] :

$$[O_{ij}, O_{kl}] = \delta_{il}O_{kj} - \delta_{kj}O_{il}$$

Let us brief up the **Properties of IBPs** :

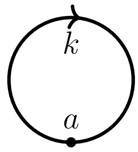
- Total number of O_{ij} operators is : $L(L+E)$ which is the number of IBPs for a given Topology.
- IBPs can only relate the Feynman integrals belonging to the same Topology tree, i.e. Integrals which can be transformed to each other by 'shrinking', as explained earlier.
- IBPs are consequence of the fact that Integral of a total derivative is zero in dimensional regularization. Another way to see it is that the dimensionally regularized Feynman Integrals must stay invariant under redefinition of loop momenta.

Let us now understand how IBPs relate Feynman Integrals through following example.

1-Loop Massive Vacuum Bubble

The Feynman Integral and Diagram corresponding to Vacuum Bubble are given as :

$$J(a) = \frac{1}{i\pi^{d/2}} \int d^d k \frac{1}{(k^2+m^2)^a} =$$



where, $D = k^2 + m^2 \quad \Rightarrow \quad k^2 = D - m^2$

as $E = 0, L = 1$, we have only 1 IBP Identity : $O_{11} = \frac{\partial}{\partial k} \cdot k$
Let us now use the IBP to obtain a recurrence relation for this Topology.

$$\begin{aligned}
&O_{11} J(a) = 0 \\
\Rightarrow &\int d^d k \frac{\partial}{\partial k^\mu} \left[\frac{k^\mu}{(k^2+m^2)^a} \right] = 0 \\
\Rightarrow &\int d^d k \left\{ \frac{d}{(k^2+m^2)^a} + k^\mu \frac{\partial}{\partial k^\mu} \left[\frac{1}{(k^2+m^2)^a} \right] \right\} = 0 \\
\Rightarrow &\int d^d k \left\{ \frac{d}{(k^2+m^2)^a} + \frac{-2a \cdot k^2}{(k^2+m^2)^{a+1}} \right\} = 0
\end{aligned} \tag{1}$$

Substitute $k^2 = D - m^2$, We get,

$$\begin{aligned}
\Rightarrow &\int d^d k \left\{ \frac{d}{D^a} + \frac{-2a \cdot (D - m^2)}{D^{a+1}} \right\} = 0 \\
\Rightarrow &\int d^d k \left\{ \frac{d - 2a}{D^a} + \frac{2am^2}{D^{a+1}} \right\} = 0 \\
\Rightarrow &\int d^d k \frac{1}{D^{a+1}} = \frac{-(d-2a)}{2am^2} \int d^d k \frac{1}{D^a} \\
\Rightarrow &\, J(a+1) = \frac{a - \frac{d}{2}}{am^2} \, J(a)
\end{aligned} \tag{2}$$

We know for $a \leq 0$, The Integral is zero because of reasons mentioned earlier in the sector analysis. Therefore, for $a > 0$ using this relation recursively we can represent every Feynman Integral of this topology in terms of $J(1)$ only. Recursion gives :

$$J(a) = \frac{\Gamma\left(a - \frac{d}{2}\right)}{m^{2(a-1)}\ \Gamma(a)\ \Gamma\left(1 - \frac{d}{2}\right)}\ J(1)$$

Therefore, Because all the Feynman Integrals of this Topology can be expressed in terms of

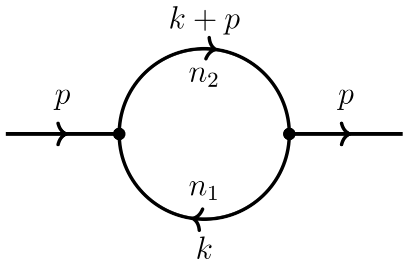
$$J(1) = \frac{1}{i\pi^{d/2}} \int d^d k \frac{1}{k^2 + m^2}$$

It is called as the Master Integral of the Vacuum Bubble Topology.

1-Loop Massive Self Energy Diagram

The Feynman Integral and Diagram corresponding to Tadpole Topology is :

$$J(n_1, n_2) = \frac{1}{i\pi^{d/2}} \int d^d k \frac{1}{D_1^{n_1} D_2^{n_2}}$$



where,

$$D_1 = k^2 + m^2 \quad \& \quad D_2 = (k+p)^2 + m^2$$

Here, we have single loop momenta and single external momenta which are $\{ k^\mu \}$ & $\{ p^\mu \}$ Therefore, The Scalar Products $k^2, k \cdot p$ can be expressed as :

$$k^2 = D_1 - m^2 \quad \& \quad k \cdot p = \frac{D_2 - D_1 - p^2}{2}$$

As $L = 1, E = 1$ we have 2 IBP Identities at our hand with Operators as follows :

$$O_{11} = \frac{\partial}{\partial k} \cdot k \quad \& \quad O_{12} = \frac{\partial}{\partial k} \cdot p$$

The O_{12} Identity gives trivial relations which can also be obtained by variable change $k \to -k-p$. Therefore, Let us solve for a relation in the all positive sector $n_1, n_2 \geq 0$ using the $O_{11} J(n_1, n_2) = 0$

$$\Rightarrow \int d^d k \frac{\partial}{\partial k^\mu} \left[\frac{k^\mu}{(k^2+m^2)^{n_1}((k+p)^2+m^2)^{n_2}} \right] = 0$$
$$\Rightarrow \int d^d k \left\{ \frac{d}{D_1^{n_1} D_2^{n_2}} + \frac{k^\mu}{D_2^{n_2}} \frac{\partial}{\partial k^\mu} \frac{1}{D_1^{n_1}} + \frac{k^\mu}{D_1^{n_1}} \frac{\partial}{\partial k^\mu} \frac{1}{D_2^{n_2}} \right\} = 0$$
$$\Rightarrow \int d^d k \left\{ \frac{d}{D_1^{n_1} D_2^{n_2}} - \frac{n_1 (2k^2)}{D_1^{n_1+1} D_2^{n_2}} - \frac{n_2 \; 2k \cdot (k+p)}{D_1^{n_1} D_2^{n_2+1}} \right\} = 0$$

Substituting,

$$k^2 = D_1 - m^2 \quad \& \quad k \cdot p = \frac{D_2 - D_1 - p^2}{2}$$

We get,

$$\int d^d k \left\{ \frac{d}{D_1^{n_1} D_2^{n_2}} - \frac{n_1 (2D_1 - 2m^2)}{D_1^{n_1+1} D_2^{n_2}} - \frac{n_2 \; (D_2 + D_1 - 2m^2 - p^2)}{D_1^{n_1} D_2^{n_2+1}} \right\} = 0$$

Simplifying results in following relation :

$$(d - 2n_1 - n_2) J(n_1, n_2) + 2m^2 n_1 J(n_1+1, n_2) - n_2 J(n_1 - 1, n_2 + 1) + n_2 (2m^2 + p^2) J(n_1, n_2 + 1) = 0$$

Let us try to understand how this relation can be used to relate Feynman Integrals. Let us put $n_1 = 1$ & $n_2 = 1$ and simplify expression to get :

$$\Rightarrow (d-3)J(1,1) + 2m^2 J(2,1) - J(0,2) + (2m^2+p^2)J(1,2) = 0$$

$$\{\because k \to -k-p \quad \Rightarrow J(n_1,n_2) = J(n_2,n_1)\}$$

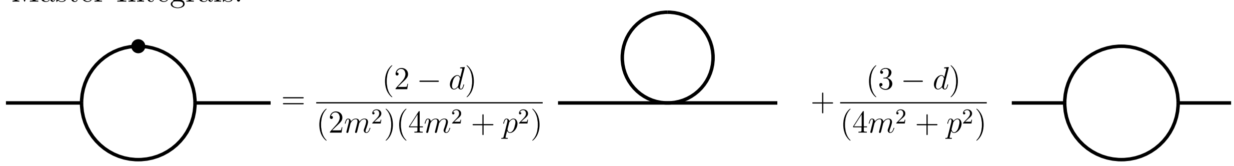
$$\Rightarrow \quad J(1,2) = \frac{1}{(4m^2+p^2)}J(0,2) - \frac{(d-3)}{(4m^2+p^2)}J(1,1)$$

However, from previous example we know,

$$\Rightarrow \quad J(0,2) = \frac{1-\frac{d}{2}}{m^2}J(0,1)$$

$$\Rightarrow \quad J(1,2) = \frac{(2-d)}{(2m^2)(4m^2+p^2)}J(0,1) - \frac{(d-3)}{(4m^2+p^2)}J(1,1)$$

This is a well known relation for the Tadpole Topology [1] and provides us great insight to identify the Master Integrals.



Now, with the help of recurrence relation obtained from the IBP Identity,we can observe that the Integral $J(1,2)$ has been completely expressed in terms $J(0,1)$ & $J(1,1)$. It can be convinced that by using this relation recursively, any Integral in the Positive sector, can be expressed solely in terms of these two Integrals. Note the role played by The symmetry of the Tadpole $J(n_1,n_2) = J(n_2,n_1)$ and the recurrence relations used in deciding the Master Integrals. Similarly, other sectors can be solved with the Master Integrals.

We just examined how the IBP Identities provide us with the necessary recurrence relations which can be used to relate various Feynman Integrals. One thing I would like to repeat again is the fact that these identities can only relate the Feynman Integrals belonging to the same Topology tree. IBPs can be used to reduce the Integrals in terms of some other Integrals, consequently help to find out the Master Integrals of the Topology. Therefore, thousands of Feynman Integrals now can be expressed in terms of a much smaller subset of Integrals called Master Integrals [4]. It can also be proved that the set of Master Integrals is finite [5]. As a result, the number of Integrals to be evaluated is reduced significantly. This process of using IBP Identities to obtain the set of Master Integrals is termed as IBP Reduction. For arbitrary powers of propagators, the algorithm for IBP Reduction need to be determined by hand. To Automate the process, method of Grobner Bases can be used [7]. The Lie-algebra structure of IBP Identities can also be exploited for the same purpose [3]. In case of evaluating with specific powers of propagators, Laporta Algorithm is widely used [8]. The algortihm follows a straightforward approach to use the specific values of the propagator powers and solve the resulting huge linear systems. `LiteRed` is one of the public implementaions of the Laporta Algortihm, which I used for my study during this project [9],[10]. I will now develop some more formalism for the IBPs and then deal with a complex problem in detail.

Operator Representation

Let us now introduce the Operators [3] A_α and B_α acting on the functions in $\mathbb{Z}^N$ as follows :

$$(A_\alpha f)(n_1,\ldots,n_\alpha,\ldots,n_N) = n_\alpha \ f(n_1,\ldots,n_\alpha+1,\ldots,n_N)$$
$$(B_\alpha f)(n_1,\ldots,n_\alpha,\ldots,n_N) = \ f(n_1,\ldots,n_\alpha-1,\ldots,n_N)$$

These operators act on the function and not on the arguments. Realize that,

$$(A_\alpha B_\alpha f)(n_1,\ldots,n_\alpha,\ldots,n_N) = n_\alpha \ f(n_1,\ldots,n_\alpha,\ldots,n_N)$$

On the other hand,

$$(B_\alpha A_\alpha f)(n_1,\ldots,n_\alpha,\ldots,n_N) = (n_\alpha - 1) \ f(n_1,\ldots,n_\alpha,\ldots,n_N)$$

Also, if $\alpha \neq \beta$ it can be observed that,

$$(A_\alpha B_\beta f)(n_1,\ldots,n_\alpha,\ldots,n_\beta,\ldots,n_N) = (B_\beta A_\alpha f) f(n_1,\ldots,n_\alpha,\ldots,n_\beta,\ldots,n_N)$$

Therefore,

The Operators A_α and B_α form the Weyl algebra and their commutator is given as :

$$[A_\alpha, B_\beta] = \delta_{\alpha\beta}$$

With the help of these operators, IBP Identities can be expressed as constraints on the function J having the form :

$$-PJ = 0$$

where,

$$P = a^{\alpha\beta} A_\alpha B_\beta + b^\alpha A_\alpha + c$$

such that $a^{\alpha\beta}$, b^α, c are just some coefficients.

For example, Let us consider the IBP Identity obtained in the last problem.

$$(d-2n_1-n_2)J(n_1,n_2) + 2m^2 n_1 J(n_1+1,n_2) - n_2 J(n_1-1,n_2+1) + n_2(2m^2+p^2)J(n_1,n_2+1) = 0$$

Let us represent it in terms of the A_α, B_α operators :

$$(d-2A_1B_1-A_2B_2)J(n_1,n_2) + 2m^2 A_1 J(n_1,n_2) - A_2B_1 J(n_1,n_2) + A_2(2m^2+p^2)J(n_1,n_2) = 0$$

$$\Rightarrow \quad -\Big[2A_1B_1 + A_2B_2 + A_2B_1 - 2m^2A_1 - (2m^2+p^2)A_2 - d\Big] J(n_1,n_2) = 0$$

Therefore, Corresponding to the IBP Identity O_{11} in the previous problem, we have a operator P_{11} in terms of the operators A_α, B_α and is given by :

$$P_{11} = 2A_1B_1 + A_2B_2 + A_2B_1 - 2m^2A_1 - (2m^2+p^2)A_2 - d$$

Similarly, for Vacuum bubble problem, we have P_{11} as :

$$P_{11} = 2A_1B_1 - 2m^2A_1 - d$$

Along with these A_α, B_α operators, we can also use the Index shifting operators. Index Shifting Operators are defined as follows :

$$\begin{aligned}
(\mathbf{a}^+ f)(n_1, \ldots, n_a, \ldots, n_N) = &\ f(n_1, \ldots, n_a + 1, \ldots, n_N) \\
(\mathbf{b}^- f)(n_1, \ldots, n_b, \ldots, n_N) = &\ f(n_1, \ldots, n_b - 1, \ldots, n_N)
\end{aligned}$$

These shift operators are inverse to each other :

$$\Rightarrow \mathbf{a}^+\mathbf{a}^- = \mathbf{a}^-\mathbf{a}^+ = \mathbf{1}$$

Easy to conclude that they all commute with each other. The earlier IBP Relations can be given in terms of Index operators as follows :

Vacuum Bubble :

$$\Rightarrow \quad \Big(d - 2n + 2n \cdot m^2 \mathbf{1}^+\Big) J(n) = 0$$

1-Loop Self Energy Diagram :

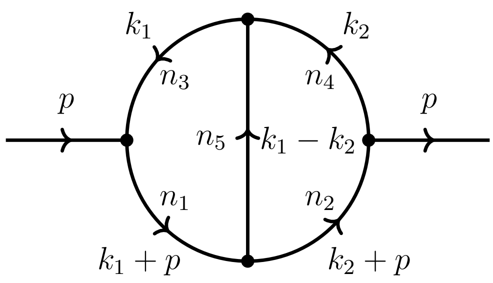
$$\Rightarrow \quad \Big(d - 2n_1 - n_2 + n_1 2m^2 \mathbf{1}^+ + n_2 \mathbf{2}^+ (2m^2 + p^2 - \mathbf{1}^-)\Big) J(n_1, n_2) = 0$$

We have had general idea of dealing with these Feynman Integrals along with application of IBPs. Now, Let us apply our concepts to a more complex problem.

2-Loop Massless Self Energy Diagram

The Feynman Integral and the Diagram corresponding to the problem are : [2]

$$J(n_1, n_2, n_3, n_4, n_5) = J(\overline{n}) = \frac{1}{(i\pi^{d/2})^2} \int d^d k_1 d^d k_2 \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}}$$



where,

$$D_1 = -(k_1 + p)^2, \quad D_2 = -(k_2 + p)^2, \quad D_3 = -k_1^2, \quad D_4 = -k_2^2, \quad D_5 = -(k_1 - k_2)^2$$

We have loop momenta as $\{k_1^\mu, k_2^\mu\}$ and external momenta as $\{p^\mu\}$, $\Rightarrow\ L = 2$ & $E = 1$. Therefore, as we can see in the integral, number of scalar products N is 5.

Because we are dealing with a Scalar Feynman Integral, it can only depend on the Lorentz scalars which do not include any loop momenta. For our problem, The Integral can depend only on p^2. By Dimensional analysis, we can write :

$$J(\overline{n}) = G(\overline{n})(-p^2)^{d-n_1-n_2-n_3-n_4-n_5}$$

For simplicity, let us set $p^2 = -1$ such that $J(\overline{n}) \Rightarrow G(\overline{n})$

Let's have a look at the symmetry of the problem now.

Consider the redefinition of loop momenta as $k_1 \to -k_1 - p$, $k_2 \to -k_2 - p$. It will result in $D_1 \leftrightarrow D_3$ & $D_2 \leftrightarrow D_4$.

Similarly, for $k_1 \to k_2$, $k_2 \to k_1$. It will result in $D_1 \leftrightarrow D_2$ & $D_3 \leftrightarrow D_4$.

Therefore, we can conclude that the Integral $J(n_1, n_2, n_3, n_4, n_5)$ is symmetric with respect to the interchanges $(1 \leftrightarrow 3, 2 \leftrightarrow 4)$ and $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$.

Observe that, if both $n_1 \leq 0$ and $n_2 \leq 0$ then, $\Rightarrow\ J(\overline{n}) = 0$, because then it becomes a scaleless Integral which is just zero in dimensional regularization.

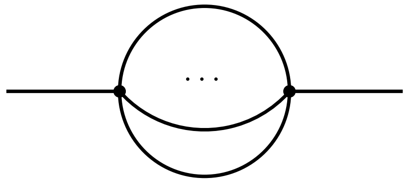
Because of the symmetry, same argument applies to $n_3 \leq 0$ and $n_4 \leq 0$.

Now, if we consider that $n_5 = 0$, it can be observed that the Integral transforms into multiplication of two one-loop Integrals. Same thing can also be concluded via the Diagram. And, now if any one of the remaining indices is zero, then we have multiplication with a scaleless Integral which ultimately results in zero.

Concluding all our arguments, we can say that the Integral $J(\overline{n})$ is zero if indices of two adjacent lines are non-positive integers, because then it contains a scaleless Integral.

Before moving on, let me first introduce the well known n-loop massless sunset Integral, which we will run across in our problem. It is given as follows :

$$G_n = \frac{\Gamma(n+1-n\frac{d}{2})\ \Gamma^{n+1}(\frac{d}{2}-1)}{\Gamma\left((n+1)(\frac{d}{2}-1)\right)} =$$



Consider the case where $n_5 = 0$ and the rest n_i are all integers. We can see that, resultant Integral is just the multiplication of two one loop massless integrals. Hence, all such Integrals are proportional to the G_1^2.

Now, consider the case where $n_1 = 0$ and the rest n_i are all integers. We will first calculate the inner loop which will shift the power of the upper propagator in the outer loop) and then calculate the outer loop. Diagrammatically it can be represented as follows :

It can be shown that all such Integrals are proportional to G_2. The cases $n_2 = 0, n_3 = 0, n_4 = 0$ are all symmetric to the $n_1 = 0$ case.

Corresponding to the loop momenta $\{k_1^\mu, k_2^\mu\}$ and external momenta $\{p^\mu\}$, we have 5 scalar products which can be expressed as :

$$k_1^2 = -D_3, \quad k_2^2 = -D_4, \quad k_1 \cdot k_2 = \frac{D_5 - D_3 - D_4}{2}$$

$$k_1 \cdot p = \frac{D_3 - D_1 + 1}{2}, \quad k_2 \cdot p = \frac{D_4 - D_2 + 1}{2}$$

And the 6 IBP Identities are given as :

$$O_{11} = \frac{\partial}{\partial k_1} \cdot k_1, \quad O_{12} = \frac{\partial}{\partial k_1} \cdot k_2, \quad O_{13} = \frac{\partial}{\partial k_1} \cdot p$$

$$O_{21} = \frac{\partial}{\partial k_2} \cdot k_1, \quad O_{22} = \frac{\partial}{\partial k_2} \cdot k_2, \quad O_{23} = \frac{\partial}{\partial k_2} \cdot p$$

Now let us consider the all positive sector with the help of IBP Identities. The $\partial_2 \cdot k_2$ relation in all positive sector can be given as follows :

$$O_{22} J(\overline{n}) = 0$$

$$\int d^dk_1 d^dk_2 \frac{\partial}{\partial k_2^\mu}\left[\frac{k_2^\mu}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}}\right] = 0$$

$$\int d^dk_1 d^dk_2 \left\{ \frac{d}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} + \frac{n_2(2k_2^2 + 2k_2 \cdot p)}{D_1^{n_1} D_2^{n_2+1} D_3^{n_3} D_4^{n_4} D_5^{n_5}} \right.$$
$$\left. + \frac{n_4(2k_2^2)}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4+1} D_5^{n_5}} + \frac{n_5(2k_2^2 - 2k_1 \cdot k_2)}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5+1}} \right\} = 0$$

Substituting the Scalar products in terms of the Denominators, simplifying and introducing the Index shifting operators we get,

$$\left[d - n_2 - n_5 - 2n_4 + n_2\mathbf{2}^+(1-\mathbf{4}^-) + n_5\mathbf{5}^+(\mathbf{3}^- - \mathbf{4}^-)\right] G(\overline{n}) = 0$$

Similarly, Let us obtain the $\partial_2 \cdot k_1$ relation :

$$O_{21}J(\overline{n}) = 0$$

$$\int d^dk_1 d^dk_2 \frac{\partial}{\partial k_2^\mu}\left[\frac{k_1^\mu}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_4^{n_4}D_5^{n_5}}\right] = 0$$

$$\int d^dk_1 d^dk_2 \left\{\frac{n_2(2k_1\cdot k_2 + 2k_1\cdot p)}{D_1^{n_1}D_2^{n_2+1}D_3^{n_3}D_4^{n_4}D_5^{n_5}} + \frac{n_4(2k_1\cdot k_2)}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_4^{n_4+1}D_5^{n_5}} + \frac{n_5(2k_1\cdot k_2 - 2k_1^2)}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_4^{n_4}D_5^{n_5+1}}\right\} = 0$$

$$\Rightarrow \left[-n_4 + n_5 + n_2\mathbf{2}^+(1+\mathbf{5}^- - \mathbf{4}^- - \mathbf{1}^-) + n_4\mathbf{4}^+(\mathbf{5}^- - \mathbf{3}^-) + n_5\mathbf{5}^+(\mathbf{3}^- - \mathbf{4}^-)\right] G(\overline{n}) = 0$$

Subtracting both equations we will obtain the $\partial_2 \cdot (k_2 - k_1)$ relation :

$$\Rightarrow \quad \left[d - n_2 - n_4 - 2n_5 + n_2\mathbf{2}^+(\mathbf{1}^- - \mathbf{5}^-) + n_4\mathbf{4}^+(\mathbf{3}^- - \mathbf{5}^-)\right] G(\overline{n}) = 0$$

Now observing this expression, we can see it contains $\mathbf{1}^-, \mathbf{3}^-, \mathbf{5}^-$. Therefore, each application of this relation will reduce $n_1 + n_3 + n_5$ by 1 corresponding to our $G(n_1, n_2, n_3, n_4, n_5)$ function. Using it recursively, sooner or later, we will come across a case where at least one of the n_1 or n_3 or n_5 is zero. As a result, the Integral can be expressed in terms of G_1^2 or G_2. Concluding all our arguments, we can confirm that any Integral in the positive sector can be expressed in terms of these G_1^2 & G_2 two Integrals, which are thus the Master Integrals.

A particular Interesting relation called as the Larin's relation can also be used for the same purpose. Let us see how to obtain that relation :

Consider our original Integral expression before we set $p^2 = -1$:

$$J(\overline{n}) = \frac{1}{(i\pi^{d/2})^2}\int d^dk_1 d^dk_2 \frac{1}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_4^{n_4}D_5^{n_5}} = G(\overline{n})(-p^2)^{d-\sum n_i}$$

Now, Let us act with $\frac{\partial}{\partial p}\cdot(k_1 + p)$ on the both sides.

Left hand side of the expression $\Rightarrow$

$$= \frac{1}{(i\pi^{d/2})^2}\int d^dk_1 d^dk_2 \frac{\partial}{\partial p^\mu}\left[\frac{(k_1+p)^\mu}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_4^{n_4}D_5^{n_5}}\right]$$
$$= \frac{1}{(i\pi^{d/2})^2}\int d^dk_1 d^dk_2 \left\{\frac{d}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_4^{n_4}D_5^{n_5}} + \frac{n_1 2(k_1+p)^2}{D_1^{n_1+1}D_2^{n_2}D_3^{n_3}D_4^{n_4}D_5^{n_5}} + \frac{n_2 2(k_2+p)\cdot(k_1+p)}{D_1^{n_1}D_2^{n_2+1}D_3^{n_3}D_4^{n_4}D_5^{n_5}}\right\}$$

Replacing scalar products with denominators, introducing index shifting operators and setting $p^2 = -1$ again we get,

$$= \left[d - 2n_1 - n_2 + n_2\mathbf{2}^+(\mathbf{5}^- - \mathbf{1}^-)\right] G(\overline{n})$$

Now, let us deal with the right hand side of our expression. But we have to be careful here to act with $\frac{\partial}{\partial p}\cdot(k_1 + p)$ on the right hand side, because it must be independent of loop momenta and thus we can't directly apply it. We will have to first express it in terms of denominator terms which are equivalent to the Index shifting operators on the right hand side. Let us see how :

$$(k_1+p)^\mu \to \frac{(k_1+p)\cdot p}{p^2} p^\mu \to \left(1+\frac{D_1-D_3}{-p^2}\right)\frac{p^\mu}{2} \to \left(1+\frac{\mathbf{1}^- - \mathbf{3}^-}{-p^2}\right)\frac{p^\mu}{2}$$

Therefore, Applying the operator on right hand side it will look like :

$$\begin{aligned}
&= \left(1+\frac{\mathbf{1}^- - \mathbf{3}^-}{-p^2}\right) G(\overline{n}) \frac{\partial}{\partial p^\mu}\left[\frac{p^\mu}{2}(-p^2)^{d-\sum n_i}\right] \\
&= \left(1+\frac{\mathbf{1}^- - \mathbf{3}^-}{-p^2}\right) G(\overline{n}) \left[(-p^2)(d-\sum n_i)(-p^2)^{d-1-\sum n_i} + \frac{d}{2}(-p^2)^{d-\sum n_i}\right]
\end{aligned}$$

Setting $p^2=-1$ we get the right hand side as :

$$\Rightarrow \quad = (\frac{3d}{2} - \sum n_i)(1+\mathbf{1}^- - \mathbf{3}^-)G(\overline{n})$$

As a result, subtracting the left hand side from right hand side, we obtain the Larin's relation [6] :

$$\Rightarrow \quad \left[\frac{d}{2} + n_1 - n_3 - n_4 - n_5 + \left(\frac{3d}{2} - \sum n_i\right)(\mathbf{1}^- - \mathbf{3}^-) + n_2\mathbf{2}^+(\mathbf{1}^- - \mathbf{5}^-)\right] G(\overline{n}) = 0$$

This expression can be used recursively to reduce Integrals in place of the earlier one.

Equivalence of IBP relations for same total number of Loop & External Momenta

IBP Relations for a generic Scalar Feynman Integral with L loop momenta and E external momenta can be reduced to have the same form as for an $M = L + E$ loop vacuum Integral. Then they differ only in the boundary conditions. Let us see how.

We know for a L-loop and Integral with E external momenta, the scalars which can be built from the momenta are :

$$s_{ij} = q_i \cdot q_j \quad i, j \in [1, M], j \geq i$$

We classify these Lorentz scalars as Scalar products and Kinematic products. Scalar Products correspond to Integration variables and their number is N. $s_{ij} : i \in [1, L]$ & $j \geq i$ correspond to the Scalar products. Kinematic products are scalars built from external momenta only and their number is N_E. $s_{ij} : i \in [L+1, L+E]$ & $j \geq i$ correspond to the Kinematic products. Total number of scalars K is thus given by :

$$K = N + N_E = \frac{M(M+1)}{2} \quad ; \; N = EL + \frac{L(L+1)}{2} \; \& \; N_E = \frac{E(E+1)}{2}$$

Now consider the Integral $J(n_1, \ldots, n_N)$ at some Kinematic point $s_{ij} = s_{ij}^0$ ($i \in [L+1, M]$ & $j \geq i$). And now introduce the quantities $D_a = -s_{ij} + s_{ij}^0 = -s_{ij} + m_a^2$ ($a \in [N+1, K]$) corresponding to every kinematic product.

Now let us re-express the Integral as :

$$J(n_1, \ldots, n_N) = D^{(E+1-d)/2} \widetilde{J}(n_1, \ldots, n_N)$$

$$D = \frac{det \; s_{ij}}{det \; s_{ij}^0} \; : \; i, j \in [L+1, M]$$

Now we expand it in powers of $D_a \; : \; a > N$.

$$\widetilde{J}(n_1, \ldots, n_N) = \sum_{n_{N+1}=1}^{\infty} \cdots \sum_{n_K=1}^{\infty} \widetilde{J}(n_1, \ldots, n_N, n_{N+1}, \ldots, n_K) D_{N+1}^{n_{N+1}-1} \cdots D_K^{n_K-1}$$

We are mainly interested in the value of our Integral at this Kinematic point ($n_{N+1} = \cdots = n_K = 1$). We assume $\widetilde{J}(n_1, \ldots, n_N, n_{N+1}, \ldots, n_K) = 0$ if any index $n_{N+1}, \ldots, n_K \leq 0$, so that there is only one sector corresponding to each of these "external" indices.

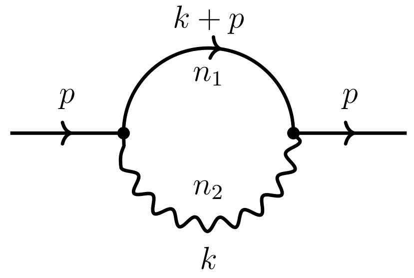
The usual IBP relations can be obtained by applying N operators O_{ij} with $i \in [1, L], j \geq i$ to the usual integrand f. Thus, the "evolution" of $\widetilde{J}(n_1, \ldots, n_N, n_{N+1}, \ldots, n_K)$ with respect to the "internal" indices $n_1, \ldots, n_N$ is governed by these N "vacuum" IBP relations.

We also need N_E additional relations which govern the "evolution" of $\widetilde{J}(n_1, \ldots, n_N, n_{N+1}, \ldots, n_K)$ with respect to the "external" indices $n_{N+1}, \ldots, n_K$. They can be obtained by applying N_E operators $O_{ij} = q_j \cdot \partial_i \; : \; i \in [L+1, M], j \geq i$ to the both sides of the expansion of $\widetilde{J}(n_1, \ldots, n_N)$ in powers of $D_a, a > N$. Both sides can be expanded in powers of D_a and the identical coefficients give us the relations with respect to "external" indices.

In this way, both "internal" and "external" relations can be obtained for $\widetilde{J}(n_1,\ldots,n_N,n_{N+1},\ldots,n_K)$ and all of them have same form as a $M = L+E$ loop vacuum Integral $J_0(n_1\ldots,n_K)$. However, the boundary conditions may differ. In the case of $\widetilde{J}(n_1,\ldots,n_N,n_{N+1},\ldots,n_K)$, for all $n_a \leq 0 \;:\; a \in [N+1,K]$, the Integral is zero. This does not have to be the case for the corresponding vacuum diagram.

Let's consider the example of 1-loop self energy Diagram (set $m^2 = 1$) :

$$M(n_1,n_2) = \frac{1}{i\pi^{d/2}}\int d^dk \frac{1}{D_1^{n_1}D_2^{n_2}}$$



where,

$$D_1 = 1-(k+p)^2 \quad \& \quad D_2 = -k^2$$

We know the Integral vanishes if $n_1 \leq 0$ because then it becomes a scaleless Integral. Here, as $L = 1, E = 1 \Rightarrow N = 2, N_E = 1$. Therefore, The Scalar products are $\{k^2, k\cdot p\}$ and Kinematics products : $\{p^2\}$

Let us consider we want to calculate it on the Kinematic point $p^2 = 1$. Consequently, we introduce $D_3 = -p^2 + 1$ corresponding to $\{p^2\}$. We can now re-express the Integral as follows :

$$M(n_1,n_2) = (p^2)^{(2-d)/2}\widetilde{M}(n_1,n_2)$$

and now expanding in powers of D_3 :

$$\widetilde{M}(n_1,n_2) = \sum_{n_3=1}^{\infty}\widetilde{M}(n_1,n_2,n_3)D_3^{n_3-1}$$

Here, $\widetilde{M}(n_1,n_2,n_3)$ vanishes if $n_3 \leq 0$. Let us calculate the usual IBP Relations which will give us evolution of $\widetilde{M}(n_1,n_2,n_3)$ with respect to "internal" indices.

The $\partial\cdot k$ relation is given by :

$$\int d^dk \frac{\partial}{\partial k^\mu}\left\{\frac{k^\mu}{[1-(k+p)^2]^{n_1}[-k^2]^{n_2}}\right\} = 0$$

$$\int d^dk \left\{\frac{d}{[1-(k+p)^2]^{n_1}[-k^2]^{n_2}} + \frac{n_1 2k\cdot(k+p)^2}{[1-(k+p)^2]^{n_1+1}[-k^2]^{n_2}} + \frac{n_2(2k^2)}{[1-(k+p)^2]^{n_1}[-k^2]^{n_2+1}}\right\} = 0$$

Substituting ,

$$2k\cdot p = D_3 + D_2 - D_1 \quad \& \quad k^2 = -D_2$$

Simplifying we get,

$$\left[d - n_1 - 2n_2 + n_1\mathbf{1}^+(D_3 - \mathbf{2}^-)\right]M(n_1,n_2) = 0$$

Similarly, solving for The $\partial \cdot (k+p)$ relation we get,

$$\left[d - 2n_1 - n_2 + 2n_1\mathbf{1}^+ + n_2\mathbf{2}^+(D_3 - 1^-)\right] M(n_1, n_2) = 0$$

However, action of D_3 on $M(n_1, n_2)$ results in :

$$D_3 M(n_1, n_2) = (p^2)^{(2-d)/2} D_3 \sum_{n_3=1}^{\infty} \widetilde{M}(n_1, n_2, n_3) D_3^{n_3-1}$$

shifting summation index $n_3 \to n_3 - 1$ it is equivalent to :

$$= (p^2)^{(2-d)/2} \sum_{n_3=1}^{\infty} \mathbf{3}^- \widetilde{M}(n_1, n_2, n_3) D_3^{n_3-1}$$

Therefore, the evolution of $\widetilde{M}(n_1, n_2, n_3)$ with respect to "internal" indices :

$$\left[d - n_1 - 2n_2 + n_1\mathbf{1}^+(\mathbf{3}^- - \mathbf{2}^-)\right] \widetilde{M}(n_1, n_2, n_3) = 0$$
$$\left[d - 2n_1 - n_2 + 2n_1\mathbf{1}^+ + n_2\mathbf{2}^+(\mathbf{3}^- - 1^-)\right] \widetilde{M}(n_1, n_2, n_3) = 0$$

To find the evolution with respect to "external" index n_3 we act with $p \cdot \frac{\partial}{\partial p}$ in the expansion of $M(n_1, n_2)$ in powers of D_3. Left hand side will look like :

$$\begin{aligned}
&= p \cdot \frac{\partial}{\partial p} M(n_1, n_2) \\
&= \frac{1}{i\pi^{d/2}} \int d^d k \; p^\mu \frac{\partial}{\partial p^\mu} \left\{ \frac{1}{[1-(k+p)]^{n_1}[-k^2]^{n_2}} \right\} \\
&= \left[-n_1 + n_1\mathbf{1}^+(2 + \mathbf{2}^- - D_3)\right] M(n_1, n_2) \\
&= (p^2)^{(2-d)/2} \sum_{n_3=1}^{\infty} \left[-n_1 + n_1\mathbf{1}^+(2 + \mathbf{2}^- - \mathbf{3}^-)\right] \widetilde{M}(n_1, n_2, n_3) D_3^{n_3-1}
\end{aligned}$$

and now on the right hand side :

$$= p \cdot \frac{\partial}{\partial p} \left\{ (p^2)^{(2-d)/2} \sum_{n_3=1}^{\infty} \widetilde{M}(n_1, n_2, n_3) D_3^{n_3-1} \right\}$$

first We differentiate them separately ,

$$p \cdot \frac{\partial}{\partial p} \; [(p^2)^{(2-d)/2}] = (2-d)(p^2)^{(2-d)/2} \sum_{n_3=1}^{\infty} \widetilde{M}(n_1, n_2, n_3)$$

$$p \cdot \frac{\partial}{\partial p} (1-p^2)^{n_3-1} = \sum_{n_3=1}^{\infty} \widetilde{M}(n_1, n_2, n_3) \cdot 2(n_3-1)(D_3^{n_3-1} - D_3^{n_3-2})$$

However, shifting the index $n_3 \to n_3 + 1$ in the second term we can write :

$$= \sum_{n_3=1}^{\infty} [2(n_3-1) - 2n_3\mathbf{3}^+] \widetilde{M}(n_1, n_2, n_3) D_3^{n_3-1}$$

Comparing coefficients of $D_3^{n_3-1}$ from both sides we get the required evolution with respect to "external" index n_3 :

$$\left[-n_1 + n_1\mathbf{1}^+(2 + \mathbf{2}^- - \mathbf{3}^-)\right] \widetilde{M}(n_1, n_2, n_3) = \left[(2-d) + 2(n_3-1) - 2n_3\mathbf{3}^+\right] \widetilde{M}(n_1, n_2, n_3)$$

$$\Rightarrow \left[d - n_1 - 2n_3 + n_1\mathbf{1}^+(2 + \mathbf{2}^- - \mathbf{3}^-) + 2n_3\mathbf{3}^+\right] \widetilde{M}(n_1, n_2, n_3) = 0$$

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