10 Lie Groups (Chapter 2) problems :

2A :

The matrix A is given as :

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

By calculating A^2 and A^3 we observe that A follows relationship $A^{m+2} = A^m$ for all positive integers m. Therefore, exponential Taylor expansion can be expressed in A and A^2 matrices only.

$$e^{i\alpha A} = \sum_{n=0}^{n=\infty} \frac{(i\alpha A)^n}{n!} = \big(\cdots\big)A + \big(\cdots\big)A^2$$

where, the coefficients for A and A^2 can be collected to identify with the Taylor expansions of hyperbolic functions.

$$\sinh x = x + \frac{x^3}{3!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \cdots$$

Finally, with all of the algebra, we can express $e^{i\alpha A}$ as folows :

$$e^{i\alpha A} = \big(i\sin\alpha\big)A + \big(\cos\alpha\big)A^2 = \begin{pmatrix} \cos\alpha & 0 & i\sin\alpha \\ 0 & 0 & 0 \\ i\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

2B :

Assuming A and B are independent of α we define,

$$f(\alpha) = e^{i\alpha A} B e^{-i\alpha A}$$

Because A obviously commutes with itself, we can write,

$$\frac{d}{d\alpha} e^{i\alpha A} = (iA)e^{i\alpha A} = e^{i\alpha A}(iA)$$

Therefore, we can now calculate

$$\frac{d}{d\alpha} f(\alpha) = i \cdot \Big(e^{i\alpha A} A B e^{-i\alpha A} - e^{i\alpha A} B A e^{-i\alpha A}\Big)$$

$$\frac{d}{d\alpha} f(\alpha) = i \cdot \Big(e^{i\alpha A}[A, B] e^{-i\alpha A}\Big)$$

If $[A, B] = B$ then we get a simple differential equation :

$$\frac{df(\alpha)}{d\alpha} = i \cdot \Big(e^{i\alpha A} B e^{-i\alpha A}\Big) = i \cdot f(\alpha)$$

Solution of which is given by,

$$f(\alpha) = Ce^{i\alpha}$$

But we know $f(0) = B$. Therefore, we can finally conclude that

$$f(\alpha) = e^{i\alpha A} B e^{-i\alpha A} = e^{i\alpha} B$$

2C :

3rd order expansion would require a bit extra hand work and lot of care, therefore I chose to skip it for now, considering I have got the gist of the expansion.

11 SU(2) (Chapter 3) problems :

3A :

It would be complicated to write the general procedure. I have understood the procedure involved in the last section (section 3.5 : J_3 values add), so I will choose to skip this for now. Besides, I feel comfortable with the well known singlet-triplet example with the irreducible representations of spin-0 and spin-1.

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

3B :

Let us first note down the handy relations of standard Pauli matrices corresponding to the SU(2) generators.

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$
$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$$
$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

From the last identity we contract the indices to derive a identity that we will use,

$$u_i v_j \sigma_i \sigma_j = u_i v_j \delta_{ij} + i u_i v_j \epsilon_{ijk}\sigma_k$$

Which in vector notation reduces to,

$$(\vec{u} \cdot \vec{\sigma})(\vec{v} \cdot \vec{\sigma}) = \vec{u} \cdot \vec{v} + i(\vec{u} \times \vec{v}) \cdot \vec{\sigma}$$

Setting $\vec{u} = \vec{v} = \hat{r}$ a unit vector we get,

$$(\hat{r} \cdot \vec{\sigma})^2 = \mathbb{1}_2$$

Therefore, it is an Identity matrix for even powers and the original matrix for odd powers. We go ahead and expand the exponential $e^{ir(\hat{r}\cdot\vec{\sigma})}$ in even and odd powers as follows :

$$e^{ir(\hat{r}\cdot\vec{\sigma})} = \mathbb{1}_2 \cdot \sum_{n=0}^{n=\infty} \frac{(ir)^{2n}}{(2n)!} + (\hat{r} \cdot \vec{\sigma}) \cdot \sum_{n=0}^{n=\infty} \frac{(ir)^{2n+1}}{(2n+1)!}$$

We can replace the sums by identifying them as the expansions of cos and sin functions respectively.

$$e^{ir(\hat{r}\cdot\vec{\sigma})} = \mathbb{1}_2 \cdot \big(\cos r\big) + (\hat{r} \cdot \vec{\sigma}) \cdot \big(i \cdot \sin r\big)$$

$$e^{i(\vec{r}\cdot\vec{\sigma})} = \mathbb{1}_2 \cdot \cos(|\vec{r}|) + i \cdot \left(\frac{\vec{r} \cdot \vec{\sigma}}{|\vec{r}|}\right) \cdot \sin(|\vec{r}|)$$

3C :

Spin-1 representation matrices for SU(2) are as follows :

$$J_1 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The structure constant of generators is ϵ_{abc} thus the adjoint representation matrices $[T_a]_{bc} = -i\epsilon_{abc}$ are given as :

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Two matrices A and B are said to be similar if there exists a non-singular matrix P responsible for the following similarity transformation :

$$A = P \cdot B \cdot P^{-1}$$

To find the P matrix, the matrices A and B should have same set of eigenvalues such that when they are diagonalized they are identical. Their Diagonalized form can be written as :

$$D = M^{-1} \cdot A \cdot M \quad \& \quad D = N^{-1} \cdot B \cdot N$$

Where, M and N are eigenvector matrices of A and B respectively. Then the P matrix responsible for similarity transformation can be obtained with a little algebra to be $P = M \cdot N^{-1}$. We can efficiently calculate all P_i matrices relating $J_i = P_i \cdot T_i \cdot P_i^{-1}$ using following Mathematica[4] code :

```
B = {{0, 0, 0}, {0, 0, -i}, {0, i, 0}};
A = 1/Sqrt[2]*{{0, 1, 0}, {1, 0, 1}, {0, 1, 0}};
vecsB = Eigenvectors[B];
vecsA = Eigenvectors[A];
P = Transpose[vecsA].Inverse[Transpose[vecsB]] // Chop // MatrixForm
```

We obtain the corresponding P matrices as follows :

$$P_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & i\sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 1 & -1 \\ -\sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad P_3 = \frac{1}{2} \cdot \begin{pmatrix} i & 1 & 0 \\ 0 & 0 & 2 \\ -i & 1 & 0 \end{pmatrix}$$

3D :

We know σ_2 and η_1 are given as follows :

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \eta_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $\sigma_2 \otimes \eta_1$ can directly be given as :

$$\sigma_2 \otimes \eta_1 = \begin{pmatrix} 0 \otimes \eta_1 & -i \otimes \eta_1 \\ i \otimes \eta_1 & 0 \otimes \eta_1 \end{pmatrix}$$

$$\sigma_2 \otimes \eta_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

3E :

Because σ and η are two different independent representations, they commute with each other. We will write the pauli matrices like matrices and use the identities to calculate.

[4] replace i with appropriate command for i in mathematica which is esc i i esc .

$$\sigma_1(\sigma_2\eta_1) = (\sigma_1\sigma_2)\eta_1 = i\sigma_3\eta_1$$
$$(\sigma_1\eta_2)(\sigma_1\eta_3) = (\sigma_1^2)(\eta_2\eta_3) = i\eta_1$$

The problem (a) is :

$$[\sigma_a, \sigma_b\eta_c] = [\sigma_a, \sigma_b]\eta_c = 2i\epsilon_{abd}\sigma_d\eta_c$$

The problem (c) is :

$$[\sigma_1\eta_1, \sigma_2\eta_2] = \sigma_1\sigma_2\eta_1\eta_2 - \sigma_2\sigma_1\eta_2\eta_1 = (i\sigma_3 \cdot i\eta_3) - (-i\sigma_3) \cdot (-i\eta_3) = 0$$

For problem (b) let's frist calculate :

$$\sigma_a\{\eta_b, \sigma_c\eta_d\} = \sigma_a\sigma_c\{\eta_b, \eta_d\} = \sigma_a\sigma_c 2\delta_{bd} = 2(\delta_{ac} + i\epsilon_{acm}\sigma_m)\delta_{bd}$$

Therefore,

$$\mathrm{Tr}\big(2(\delta_{ac} + i\epsilon_{acm}\sigma_m)\delta_{bd}\big) = 2 \cdot \mathrm{Tr}\big(\delta_{ac}\delta_{bd}\mathbb{1}_2\big) = 4\delta_{ac}\delta_{bd}$$

12 Tensor Operators (Chapter 4) problems :

First, I will note down all the different formulas which are relevant enough to refer back.

$$J^+ |j,m\rangle = \sqrt{\frac{(j-m)(j+m+1)}{2}} |j,m+1\rangle$$
$$J^- |j,m\rangle = \sqrt{\frac{(j+m)(j-m+1)}{2}} |j,m-1\rangle$$

We first write down the states for $1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$ with highest weight decomposition and acting with J^- repeatedly, and then taking the orthonormal states.

$$\left|\frac{3}{2},\frac{3}{2}\right\rangle = |1,1\rangle \left|\frac{1}{2},\frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2},\frac{1}{2}\right\rangle = \frac{\sqrt{2}}{\sqrt{3}} \cdot |1,0\rangle \left|\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}} \cdot |1,1\rangle \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}} \cdot |1,-1\rangle \left|\frac{1}{2},\frac{1}{2}\right\rangle + \frac{\sqrt{2}}{\sqrt{3}} \cdot |1,0\rangle \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$\left|\frac{3}{2},-\frac{3}{2}\right\rangle = |1,-1\rangle \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}} \cdot |1,0\rangle \left|\frac{1}{2},\frac{1}{2}\right\rangle - \frac{\sqrt{2}}{\sqrt{3}} \cdot |1,1\rangle \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \frac{\sqrt{2}}{\sqrt{3}} \cdot |1,-1\rangle \left|\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}} \cdot |1,0\rangle \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

spin 1/2 representation of the SU(2) generators :

$$J_1 = \frac{1}{2} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{2} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad J_3 = \frac{1}{2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$J^+ = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J^- = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

spin 1 representation of the SU(2) generators :

$$J_1 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$J^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad J^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Tensorial position operators in spin 1 representation : $\{r_{+1}, r_0, r_{-1}\}$ in terms of the position operators : $\{r_1, r_2, r_3\}$

$$r_{+1} = \frac{-r_1 - ir_2}{\sqrt{2}}$$

$$r_0 = r_3$$
$$r_{-1} = \frac{r_1 - ir_2}{\sqrt{2}}$$

Using the algebra $[J_a, r_b] = -i\epsilon_{acb} r_c$, we can go ahead and derive the above relationa, such that they are constrained with following algebra : (General Tensor operator transformation)

$$[J_a, O^s_l] = O^s_m [J^s_a]_{ml}$$

Let us also note down a few handy commutation relations :

$$[J^+, r_{+1}] = 0 \qquad [J^-, r_{+1}] = r_0$$
$$[J^+, r_0] = r_{+1} \qquad [J^-, r_0] = r_{-1}$$
$$[J^+, r_{-1}] = r_0 \qquad [J^-, r_{-1}] = 0$$

Finally note down the Wigner-Eckart formula to help solve problems like the problem 4A.

$$\langle J, m', \beta | \, O^s_l \, | j, m, \alpha \rangle = \delta_{m', l+m} \cdot \Big(\langle J, l+m | \cdot | s, l \rangle \, | j, m \rangle \Big) \cdot k_{\alpha\beta}$$

Where $k_{\alpha\beta} = \langle J, \beta | \, O^s \, | j, \alpha \rangle$ are reduced matrix elements, and the quantity inside the round brackets is the Clebsch-Gordan Coefficient. Let us see how to the above formula and other identities to solve the textbook problem. First it is given that

$$\left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| \, r_3 \, \middle| \frac{1}{2}, \frac{1}{2}, \beta \right\rangle = A$$
$$\implies \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| \, r_0 \, \middle| \frac{1}{2}, \frac{1}{2}, \beta \right\rangle = A$$

To calculate $\left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| \, r_1 \, \middle| \frac{1}{2}, -\frac{1}{2}, \beta \right\rangle$, we first need to express it in terms of tensorial operators to be able to use the Wigner-Eckart formula. Therefore, we calculate and substitute $r_1 = \frac{-r_{+1} + r_{-1}}{\sqrt{2}}$. However, note that $[J^-, r_0] = r_{-1}$. Thus, contribution from r_{-1} is zero as J^- kills both of the states on either left or right.

$$\implies \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| \, r_1 \, \middle| \frac{1}{2}, -\frac{1}{2}, \beta \right\rangle = \frac{-1}{\sqrt{2}} \cdot \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| \, r_{+1} \, \middle| \frac{1}{2}, -\frac{1}{2}, \beta \right\rangle$$

Now let us use the Wigner-Eckart formula on both of the matrix elements of interest.

$$A = \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| \, r_0 \, \middle| \frac{1}{2}, \frac{1}{2}, \beta \right\rangle = \left(\left\langle \frac{1}{2}, \frac{1}{2} \middle| \cdot |1, 0\rangle \middle| \frac{1}{2}, \frac{1}{2} \right\rangle \right) \cdot k_{\beta\alpha}$$
$$? = \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| \, r_{+1} \, \middle| \frac{1}{2}, -\frac{1}{2}, \beta \right\rangle = \left(\left\langle \frac{1}{2}, \frac{1}{2} \middle| \cdot |1, 1\rangle \middle| \frac{1}{2}, -\frac{1}{2} \right\rangle \right) \cdot k_{\beta\alpha}$$

The corresponding Clebsch-Gordan coefficients can be noted down from the above $1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$ decomposition. Taking the ratio, we finally obtain

$$\implies \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| \, r_{+1} \, \middle| \frac{1}{2}, -\frac{1}{2}, \beta \right\rangle = -A\sqrt{2}$$

Therefore substituting back to get the required answer,

$$\implies \left\langle \frac{1}{2}, \frac{1}{2}, \alpha \middle| \, r_1 \, \middle| \frac{1}{2}, -\frac{1}{2}, \beta \right\rangle = A$$

4A :

Given is a spin 1/2 operator with transformation :

$$[J_a, O_x] = \frac{O_y}{2} \cdot [\sigma_a]_{yx}$$

with $x = 1, 2$ and σ_a are the Pauli matrices. First let us note down the handy Commutation relations that follow from this :

$$[J^+, O_1] = 0 \qquad [J^+, O_2] = \frac{O_1}{\sqrt{2}}$$

$$[J^-, O_1] = \frac{O_2}{\sqrt{2}} \qquad [J^-, O_2] = 0$$

$$[J_3, O_1] = \frac{O_1}{2} \qquad [J_3, O_2] = -\frac{O_2}{2}$$

Actually, this problem is rather easy to just solve directly, so we will first do it directly without Tensorial operator description. Later, I will also do it by identifying the Tensorial operators. Given :

$$\langle 3/2, -1/2, \alpha| O_1 |1, -1, \beta\rangle = A$$

and we need to find

$$\langle 3/2, -3/2, \alpha| O_2 |1, -1, \beta\rangle = ?$$

We can use $O_2 = \sqrt{2} \cdot [J^-, O_1] = \sqrt{2} \cdot (J^- O_1 - O_1 J^-)$. However, observe that J^- kills the state on right. We have only the first term $\sqrt{2} J^- O_1$, and we make the J^- act on left state, which is equivalent to its hermitian conjugate acting on right, which will be

$$J^+ |3/2, -3/2, \alpha\rangle = \frac{\sqrt{3}}{\sqrt{2}} |3/2, -1/2, \alpha\rangle$$

Taking the hermitian conjugate we can write

$$\implies \langle 3/2, -1/2, \alpha| O_2 |1, -1, \beta\rangle = \sqrt{3} \cdot \langle 3/2, -1/2, \alpha| O_1 |1, -1, \beta\rangle = A\sqrt{3}$$

Which is our required answer.

Or, one can also identify the required spin 1/2 Tensorial Operators, as some linear combinations of O_1 and O_2. Turns out[5], the Operators are already in the standard tensorial description form[6]. We can identify $O_1 = O_{+1/2}$ and $O_2 = O_{-1/2}$ as they follow the exact commutation algebra as the Tensorial operators $O_{+1/2}, O_{-1/2}$ should. We can use the Wigner-Eckart theorem now for the matrix elements of interest.

$$A = \langle 3/2, -1/2, \alpha| O_{+1/2} |1, -1, \beta\rangle = \Big(\langle 3/2, -1/2| \cdot |1/2, 1/2\rangle |1, -1\rangle \Big) \cdot k_{\alpha\beta} = \frac{1}{\sqrt{3}} \cdot k_{\alpha\beta}$$

$$? = \langle 3/2, -3/2, \alpha| O_{-1/2} |1, -1, \beta\rangle = \Big(\langle 3/2, -3/2| \cdot |1/2, -1/2\rangle |1, -1\rangle \Big) \cdot k_{\alpha\beta} = k_{\alpha\beta}$$

Where we obtained the Clebsch-Gordan coefficients from the earlier defined $1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$ decomposition. Now we just take ratios to find

$$\implies \langle 3/2, -3/2, \alpha| O_{-1/2} |1, -1, \beta\rangle = A\sqrt{3}$$

Which is consistent with our earlier answer.

[5] Not really turns out, it is obvious because Pauli matrices are the spin 1/2 representation of J_a.

[6] Which is why I guess it was easier to just solve directly.

4B :

Given : $[J^+, (r_{+1})^2] = 0$

spin 2 representation of the SU(2) generators :

$$J^+ = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad J^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

We know a spin 2 tensor operator should satisfy following transformation :

$$[J^-, O_{+2}] = O_{+1}\sqrt{2}$$
$$[J^-, O_{+1}] = O_0\sqrt{3}$$
$$[J^-, O_0] = O_{-1}\sqrt{3}$$
$$[J^-, O_{-1}] = O_{-2}\sqrt{2}$$
$$[J^-, O_{-2}] = 0$$

Commuting with J^- and using $[A, BC] = [A, B]C + B[A, C]$ recursively, we can derive following relations starting from $(r_{+1})^2$:

$$[J^-, r_{+1}r_{+1}] = r_0r_{+1} + r_{+1}r_0$$
$$[J^-, r_0r_{+1} + r_{+1}r_0] = r_{-1}r_{+1} + 2(r_0)^2 + r_{+1}r_{-1}$$
$$[J^-, r_{-1}r_{+1} + 2(r_0)^2 + r_{+1}r_{-1}] = 3r_0r_{-1} + 3r_{-1}r_0$$
$$[J^-, 3r_0r_{-1} + 3r_{-1}r_0] = 6(r_{-1})^2$$
$$[J^-, 6(r_{-1})^2] = 0$$

By comparing with transformation relations for O_m operators, we can easily establish following relations :

$$O_{+2} = (r_{+1})^2$$
$$O_{+1} = \frac{r_0r_{+1} + r_{+1}r_0}{\sqrt{2}}$$
$$O_0 = \frac{r_{-1}r_{+1} + 2(r_0)^2 + r_{+1}r_{-1}}{\sqrt{6}}$$
$$O_{-1} = \frac{r_0r_{-1} + r_{-1}r_0}{\sqrt{2}}$$
$$O_{-2} = (r_{-1})^2$$

Now if we set $r_1 = \sin(\theta)\cos(\phi)$, $r_2 = \sin(\theta)\sin(\phi)$, $r_3 = \cos(\theta)$

$$\implies r_{+1} = \frac{-\sin(\theta)e^{i\phi}}{\sqrt{2}} \quad r_{-1} = \frac{\sin(\theta)e^{-i\phi}}{\sqrt{2}} \quad r_0 = \cos(\theta)$$

Then we can clearly identify our O_m operators with spherical harmonics $Y_l^m(\theta, \phi)$ with $l = 2$, with the following relation :

$$Y_2^m(\theta, \phi) = \left(\frac{1}{2} \cdot \sqrt{\frac{15}{2\pi}}\right) \cdot O_m$$

possible generalized conclusion : spin j tensor operators constructed with products of spin 1 position tensor operators, are proportional to spin j spherical harmonics.

4C :

This problem is similar to the problem 3B . In place of the Identity matrix, we need to input the said projection operator. Remember that for any projection operator P, it follows $P^n = P$, [7] for all positive integers n. We can write the format for final answer as :

$$e^{i\alpha \cdot \hat{\alpha_a} X_a} = \mathbb{1} + [\cos(\alpha) - 1] \cdot (\hat{\alpha_a} X_a)^2 + i\sin(\alpha) \cdot (\hat{\alpha_a} X_a)$$

[7] Once you project onto the system, consequent identical projections will be redundant

13 Roots and weights (Chapter 6) problems :

6A :

Using $[H_i, E_\alpha] = \alpha_i E_\alpha$ and $[H_i, E_\beta] = \beta_i E_\beta$, we can calculate the direct commutator with the cartan generator H_i

$$\begin{aligned}[H_i, [E_\alpha, E_\beta]] &= [[H_i, E_\alpha], E_\beta] + [E_\alpha, [H_i, E_\beta]] \\ &= \alpha_i [E_\alpha, E_\beta] + \beta_i [E_\alpha, E_\beta] \\ &= (\alpha+\beta)_i [E_\alpha, E_\beta] \\ [H_i, [E_\alpha, E_\beta]] &= (\alpha+\beta)_i [E_\alpha, E_\beta]\end{aligned}$$

Therefore, we can conclude that $[E_\alpha, E_\beta] \sim E_{\alpha+\beta}$

$$\implies [E_\alpha, E_\beta] = N \cdot E_{\alpha+\beta}$$

If $\alpha + \beta$ is not a root, we won't be able to conclude the same. ($N = 0$?)

6B :

Similar to problem 5A, we can prove that $[E_\alpha, E_{-\alpha-\beta}] \sim E_{-\beta}$ and $[E_\beta, E_{-\alpha-\beta}] \sim E_{-\alpha}$. Let's assume following :

$$\begin{aligned}[E_\alpha, E_{-\alpha-\beta}] &= P \cdot E_{-\beta} \\ [E_\beta, E_{-\alpha-\beta}] &= Q \cdot E_{-\alpha} \\ [E_\alpha, E_\beta] &= N \cdot E_{\alpha+\beta}\end{aligned}$$

Let's apply the Jacobi Identity to $E_\alpha, E_\beta, E_{-\alpha-\beta}$: (Also use $[E_\gamma, E_{-\gamma}] = \gamma_i H_i$

$$\begin{aligned}0 &= [E_\alpha, [E_\beta, E_{-\alpha-\beta}]] + [E_{-\alpha-\beta}, [E_\alpha, E_\beta]] + [E_\beta, [E_{-\alpha-\beta}, E_\alpha]] \\ 0 &= Q \cdot [E_\alpha, E_{-\alpha}] - N \cdot [E_{\alpha+\beta}, E_{-\alpha-\beta}] - P \cdot [E_\beta, E_{-\beta}] \\ 0 &= (Q\alpha_i - N\alpha_i - N\beta_i - P\beta_i) H_i \\ 0 &= \big((Q-N)\alpha_i + (-P-N)\beta_i\big) H_i\end{aligned}$$

If α, β are linearly independent roots, then for whole quantity to be zero, their coefficients must also be zero. (H_i form a linear space)

$$\implies Q = N \quad P = -N$$

6C :

We have $H_1 = \sigma_3 = \sigma_3 \otimes \mathbb{1}_2$ and $H_2 = \sigma_3 \otimes \tau_3$, which we can evaluate as : (Refer problem 3D)

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The corresponding eigenvectors and weights are :

$$(1,0,0,0)^T \quad : (1,1)$$

$$(0,1,0,0)^T \quad : (1,-1)$$
$$(0,0,1,0)^T \quad : (-1,-1)$$
$$(0,0,0,1)^T \quad : (-1,1)$$

We will have two zero root vectors corresponding to Cartan generators. Rest roots are obtained as differences of the above weights. Ex. $(1,1)-(1,-1)=(0,2)$ and $(1,-1)-(1,1)=(0,-2)$ and so on.

$$(0,0),\ (0,0),\ (0,2),\ (0,-2),\ (2,2),\ (-2,-2),\ (2,0),\ (-2,0),\ (2,-2),\ (-2,2)$$

14 SU(3) (Chapter 7) problems :

7A :

The SU(3) generators in fundamental representation are given as $T_a = \frac{1}{2} \cdot \lambda_a$, where λ_a are the standard 8 Gell-Mann matrices. In fundamental representation :

$$\mathrm{Tr}(T_a T_b) = \frac{1}{2} \cdot \delta_{ab}$$

and thus the real anti-symmetric structure constants defining the SU(3) algebra $[T_a, T_b] = i f_{abc} T_c$ are given as :

$$f_{abc} = -2i \cdot \mathrm{Tr}([T_a, T_b] T_c)$$

The adjoint representation is then given as :

$$(F_a)_{bc} = -i \cdot f_{abc}$$

With the standard matrices, one can easily calculate $f_{147} = \frac{1}{2}$ and $f_{458} = \frac{\sqrt{3}}{2}$.

7B :

We know the SU(2) algebra is given by $[J_i, J_j] = i\epsilon_{ijk} J_k$. In the SU(3) algebra, we can find out that the structure constant $f_{123} = f_{231} = f_{312} = +1$ and thus proving that the fundamental generators T_1, T_2, T_3 form the SU(2) sub-algebra. Specifically, one can verify that :

$$[T_1, T_2] = iT_3 \quad [T_2, T_3] = iT_1 \quad [T_3, T_1] = iT_2$$

proving the first part.

Now, observe that the these threee T_1, T_2, T_3 SU(3) generators can be written as :

$$T_a = \begin{pmatrix} S_a & 0 \\ 0 & 0 \end{pmatrix}$$

where the S_1, S_2, S_3 are the fundamental generators[8] of SU(2). Naturally, we can see that in this SU(2) sub-algebra, SU(3) triplet (3-vector) will transform as SU(2) doublet and as SU(2) singlet. S_a will act on upper two components, just like a spinor while the third component is left untouched. Refer to the formula from problem 4C to explicitely check this :

$$e^{i\alpha \cdot \hat{\alpha_a} X_a} = \mathbb{1} + [\cos(\alpha) - 1] \cdot (\hat{\alpha_a} X_a)^2 + i \sin(\alpha) \cdot (\hat{\alpha_a} X_a)$$

. The mathematical way to write this will be :

$$3 = 2 \oplus 1$$

Which in words can be stated as : Under this SU(2) sub-algebra, 3 representation of SU(3) (fundamental) can be decomposed to ir-reps of SU(2) as a direct sum of the doublet (2 or fundamental) and singlet representations of SU(2). Alternatively, one can say, under this SU(2) sub-algebra, SU(3) triplet transforms like a doublet plus singlet under SU(2). Now, Let's turn attention to transformation under adjoint representation (8 representation) of SU(3). We know that for adjoint representation we can write :

$$3 \otimes \overline{3} = 8 \oplus 1$$

[8] Generators in fundamental representation.

But, under this SU(2) sub-algebra, we know the decomposition $3 \rightarrow 2 \oplus 1$ and $\overline{3} \rightarrow 2 \oplus 1$. $(2 = \overline{2})$. Substituting and calculating the tensor product :

$$(2 \oplus 1) \otimes (2 \oplus 1) = (2 \otimes 2) \oplus (2 \otimes 1) \oplus (1 \otimes 2) \oplus (1 \otimes 1)$$

And using the adjoint representation (3) of SU(2) : $2 \otimes 2 = 3 \oplus 1$, we can write :

$$8 \oplus 1 = 3 \oplus 2 \oplus 2 \oplus 1 \oplus 1$$

Therefore, decomposition for adjoint representation under this sub-algebra is :

$$8 \rightarrow 3 \oplus 2 \oplus 2 \oplus 1$$

Stating : adjopint representaion of SU(3) is decomposed down to ir-reps of SU(2) as direct sum of 3,2,2,1 representations. Transforming like triplet plus two doublets plus a singlet.

7C :

I can prove that $\lambda_2, \lambda_5, \lambda_7$ form the SU(2) algebra with a straight-forward approach, similar to last problem. However, I am confused about the particular decomposition under this SU(2) sub-algebra. I tried infinitesimal expansion but I couldn't pick up the correct decomposition. I could use a discussion. Here all three components are mixing so I was suspecting the decomposition might as well be just $3 \rightarrow 3$ and then $8 \rightarrow 3 \oplus 5$.