Numerical Evaluation of The $\mathcal{M}_{gg\to gH}$ Amplitude : Higgs + Jet Production through Gluon Fusion

Chaitanya Paranjape - IIT(ISM) Dhanbad, India

Abstract

This is the technical project report on the numerical evaluation of the amplitude for the $gg \to gH$ process at 1-Loop. In this work, I will demonstrate the application of two new features `sum_package` and `Expansion by regions` introduced to pySecDec in [4]. The `sum_package` will be used to calculate form factors and then the amplitude, expressed as weighted sums of Master integrals, with error bound on the sum rather than individual integral. The concerned example is associated with the cross-section estimates for the Higgs + jet production at the LHC. In line with this work, multi-loop calculations can seek the application of these features.

Contents

1 Introduction

The aim of this project is to numerically evaluate the 1-Loop squared matrix Element for the process : $gg \to gH$, using the newly developed *sum_package* of `pySecDec` [4]. More about pySecDec can be found out from pySecDec [2],[3] (https://secdec.readthedocs.io/en/latest/). The *sum_package* is designed to numerically evaluate the weighted superposition of a set of feynman integrals. Therefore, our purpose would be to express the required amplitude in terms of the so called master integrals [1]. The derivation for our amplitude can be found out from [1]. Most of the equations we will be using on-wards have been borrowed from the same reference only.

In the framework of Conventional Dimensional Regularization(CDR), spin and color averaged absolute square of our amplitude is given by :

$$|\overline{\mathcal{M}_{gg\to gH}}|^2 = \frac{1}{(d-2)^2} \cdot \frac{1}{(N_c^2-1)^2} \cdot |\mathcal{M}_{gg\to gH}|^2 \tag{II.46}$$

Where we have divided by the number of polarization $(d-2)$ & the number of colors (N_c^2-1) for each of the incoming gluon. The absolute square of the matrix element is expressed in terms of the form factors [2].

$$\begin{aligned}|\mathcal{M}_{gg\to gH}|^2 = \frac{1}{4} \cdot \Big[&\frac{(d-2)|F_{212}|^2 s_{12}^3 s_{23}}{s_{13}} + F_{212}F_{332}^* s_{12}s_{23}^2 + F_{212}F_{311}^* s_{12}^2 s_{13} + (d-2)F_{212}F_{312}^* s_{12}^2 s_{23} \\ &+ F_{332}F_{212}^* s_{12}s_{23}^2 + \frac{(d-2)|F_{332}|^2 s_{13}s_{23}^3}{s_{12}} + F_{332}F_{311}^* s_{13}^2 s_{23} + (d-2)F_{332}F_{312}^* s_{13}s_{23}^2 \\ &+ F_{311}F_{212}^* s_{12}^2 s_{13} + F_{311}F_{332}^* s_{13}^2 s_{23} + \frac{(d-2)|F_{311}|^2 s_{12}s_{13}^3}{s_{23}} + (d-2)F_{311}F_{312}^* s_{12}s_{13}^2 \\ &+ (d-2)F_{312}F_{212}^* s_{12}^2 s_{23} + (d-2)F_{312}F_{332}^* s_{13}s_{23}^2 + (d-2)F_{312}F_{311}^* s_{12}s_{13}^2 \\ &+ (3d-8)|F_{312}|^2 s_{12}s_{13}s_{23} \Big]\end{aligned} \tag{II.48}$$

The form factor definitions can be obtained from [1]. Note here that only 4 of the form factors contribute to the physical amplitude. Our job would be to express these form factors in terms of the master integrals, such that we can provide them to the pySecDec for the numerical evaluation using the *sum_package*. We can then evaluate the required matrix element squared and verify our results with the analytic expressions in the Heavy Top quark Limit (HTL).

The project report has been divided into two parts, first dealing with the calculations of integrals, form factors and setting up all the equations we will be using. Second part will be focusing on the implementation of the same with pySecDec and verifying our results with the previously set up analytic expressions. We will now start with the first part by identifying the master integrals in the 1-Loop-Box topology of our $gg \to gH$ process.

All the files required for implementing our example in pySecDec can be found in the gggH1L folder at the Git-hub repository for this project : PSD_gggH1L [3]

---

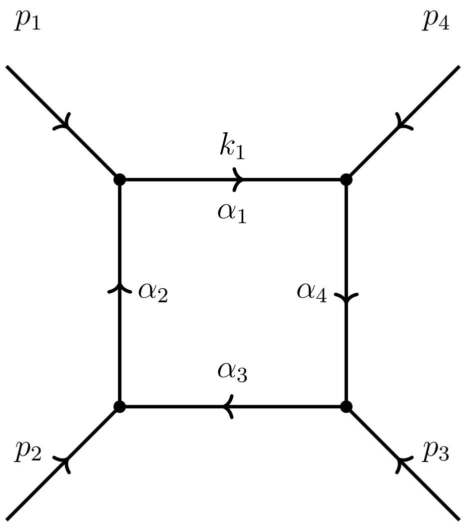
[1]The analytic calculation of the amplitude involves evaluating a large number of scalar feynman integrals. Instead of evaluating all of the integrals, we can find out relations between these integrals with the help of the Integration By Parts(IBP) identities. Using this method of IBP Reduction, we can express all the integrals in terms of a finite set of integrals called as the master integrals.

[2]The form factors are the Lorentz invariant coefficients defined in the tensorial Decomposition of the amplitude, the definitions for which can be found out from [1].

[3]https://github.com/CpSquared/PSD_gggH1L

2 The Master Integrals

All of the master integrals in our case correspond to the different sub-topology of the 1-Loop-Box topology. Let us set up the 1-Loop-Box topology for the $gg \to gH$ process.



The kinematic constraints imposed will be $p_1^2 = p_2^2 = p_3^2 = 0$ & $p_4^2 = m_H^2$ along with the momentum conservation $: p_1 + p_2 + p_3 + p_4 = 0 \;\Rightarrow\; m_H^2 = s_{12} + s_{13} + s_{23}$.

Where we have defined the kinematic in-variants as : $s_{ij} = (p_i + p_j)^2$.

Conveniently, the number of independent scalar products is same as the number of propagators. As a result, we can immediately define the general scalar feynman integral as follows :

$$I_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} = \int \frac{d^d k_1}{i\pi^{d/2}} \cdot \frac{1}{D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} D_4^{\alpha_4}}$$

with the propagators given by :

$$D_1 = k_1^2 - m_t^2$$
$$D_2 = (k_1 - p_1)^2 - m_t^2$$
$$D_3 = (k_1 - p_1 - p_2)^2 - m_t^2$$
$$D_4 = (k_1 - p_1 - p_2 - p_3)^2 - m_t^2$$

Topology of the integral depends on the propagator powers, therefore, we can obtain the different sub-topology by shrinking down the propagators, which would be setting the corresponding propagator power to zero. Key thing to note here is that, shrinking the propagators does not violate the momentum conservation at any point and the integral still belongs to the same topology family. As a result, they can still be related by the IBP identities. Which is how we reduce the set of integrals down to the set of master integrals. [4]

The master integrals for our example have already been well defined and can be found out from [1]. The 9 master integrals are as follows : ($*$ have crossed Integrals)

$$g_1(m_t^2) = I_{2,0,0,0} \qquad * \; g_2(s_{12}, m_t^2) = I_{2,0,1,0} \qquad g_3(s_{23}, m_t^2) = I_{0,2,0,1}$$
$$g_4(m_H^2, m_t^2) = I_{2,0,0,1} \qquad * \; g_5(s_{12}, m_t^2) = I_{1,1,1,0} \qquad g_6(s_{23}, m_t^2) = I_{0,1,1,1}$$
$$* \; g_7(s_{12}, m_H^2, m_t^2) = I_{1,0,1,1} \qquad g_8(s_{23}, m_H^2, m_t^2) = I_{1,1,0,1} \qquad * \; g_9(s_{12}, s_{23}, m_H^2, m_t^2) = I_{1,1,1,1}$$

[4]For more information about master Integrals and IBP reduction, kindly refer to [5].

Along with these 9 master integrals, some crossed-integrals also appear in the expressions of form-factors and amplitude. There are 5 extra crossed-integrals which we will also need in our expressions. The required 5 crossed integrals are obtained as :

- $g_2(s_{13}, m_t^2)$: obtained from $g_2(s_{12}, m_t^2)$ by Exchanging p_2 and p_3 .
- $g_5(s_{13}, m_t^2)$: obtained from $g_5(s_{12}, m_t^2)$ by Exchanging p_2 and p_3 .
- $g_7(s_{13}, m_H^2, m_t^2)$: obtained from $g_7(s_{12}, m_H^2, m_t^2)$ by Exchanging p_2 and p_3 .
- $g_9(s_{12}, s_{13}, m_H^2, m_t^2)$: obtained from $g_9(s_{12}, s_{23}, m_H^2, m_t^2)$ by Exchanging p_1 and p_2 .
- $g_9(s_{23}, s_{13}, m_H^2, m_t^2)$: obtained from $g_9(s_{12}, s_{23}, m_H^2, m_t^2)$ by Exchanging p_2 and p_3 .

Now, we have defined all the integrals we will need. For convenience, we will set a fixed order for these integrals, and will stick to this order for all the labelling in the calculations and pySecDec implementaion. The ordered 14 integrals are :

1. $g_1(m_t^2) = I_{2,0,0,0}$
2. $g_2(s_{12}, m_t^2) = I_{2,0,1,0}$
3. $g_2(s_{13}, m_t^2) = I_{2,0,1,0}$: $(p_2 \leftrightarrow p_3)$
4. $g_3(s_{23}, m_t^2) = I_{0,2,0,1}$
5. $g_4(m_H^2, m_t^2) = I_{2,0,0,1}$
6. $g_5(s_{12}, m_t^2) = I_{1,1,1,0}$
7. $g_5(s_{13}, m_t^2) = I_{1,1,1,0}$: $(p_2 \leftrightarrow p_3)$
8. $g_6(s_{23}, m_t^2) = I_{0,1,1,1}$
9. $g_7(s_{12}, m_H^2, m_t^2) = I_{1,0,1,1}$
10. $g_7(s_{13}, m_H^2, m_t^2) = I_{1,0,1,1}$: $(p_2 \leftrightarrow p_3)$
11. $g_8(s_{23}, m_H^2, m_t^2) = I_{1,1,0,1}$
12. $g_9(s_{12}, s_{23}, m_H^2, m_t^2) = I_{1,1,1,1}$
13. $g_9(s_{12}, s_{13}, m_H^2, m_t^2) = I_{1,1,1,1}$: $(p_1 \leftrightarrow p_2)$
14. $g_9(s_{23}, s_{13}, m_H^2, m_t^2) = I_{1,1,1,1}$: $(p_2 \leftrightarrow p_3)$

Now we can turn our attention to the 4 form factors and express them as superposition of the above integrals.

3 The Form Factors

There are only 4 form factors which appear in the physical amplitude, namely $F_{212}, F_{311}, F_{332}$ & F_{312}. The expressions and derivations for which can be obtained from [1]. Our task would be to simplify the provided definitions and express them as superposition of the 14 Integrals.

Out of the 4 factors : $F_{212}, F_{311}, F_{332}$ are related through the permutation in-variance of the amplitude. $F_{212}(s_{12}, s_{13}, s_{23}) = F_{311}(s_{13}, s_{23}, s_{12}) = F_{332}(s_{23}, s_{12}, s_{13})$. So, once we have F_{212} ready, pySecDec can calculate the other two easily by just changing the arguments.

Therefore, F_{212} and F_{312} are the two independent form factors we need to deal with.

Let's first focus on F_{212} :

We define temporary set of coefficients $T_1, T_2, \cdots, T_{13}$ [5] such that the F_{212} can be expressed as follows : (defined at Pg 97 of [1])

$$
\begin{aligned}
F_{212}(s_{12}, s_{13}, s_{23}) = f^{abc} C_\epsilon \cdot 2 m_t e g_{ht} g_s^3 \cdot \Big[& T_1 \cdot A(m_H^2, m_t^2) \\
& + T_2 \cdot B(s_{12}, m_H^2, m_t^2) \\
& + T_3 \cdot B(s_{13}, m_H^2, m_t^2) \\
& + T_4 \cdot B(s_{23}, m_H^2, m_t^2) \\
& + T_5 \cdot C(s_{12}, s_{13}, s_{23}, m_t^2) \\
& + T_6 \cdot C(s_{13}, s_{23}, s_{12}, m_t^2) \\
& + T_7 \cdot C(s_{23}, s_{12}, s_{13}, m_t^2) \\
& + T_8 \cdot g_7(s_{12}, m_H^2, m_t^2) \\
& + T_9 \cdot g_7(s_{13}, m_H^2, m_t^2) \\
& + T_{10} \cdot g_8(s_{23}, m_H^2, m_t^2) \\
& + T_{11} \cdot g_9(s_{12}, s_{23}, m_H^2, m_t^2) \\
& + T_{12} \cdot g_9(s_{12}, s_{13}, m_H^2, m_t^2) \\
& + T_{13} \cdot g_9(s_{23}, s_{13}, m_H^2, m_t^2) \Big] \\
\Rightarrow F_{212}(s_{12}, s_{13}, s_{23}) = f^{abc} C_\epsilon \cdot 2 m_t e g_{ht} g_s^3 \cdot & f_{212}(s_{12}, s_{13}, s_{23})
\end{aligned}
$$

The definitions for $A(), B()$ and $C()$ functions [1] are as follows :

$$
\begin{aligned}
A(m_H^2, m_t^2) &= g_1(m_t^2) + (4m_t^2 - m_H^2) g_4(m_H^2, m_t^2) \\
B(x, m_H^2, m_t^2) &= (4m_t^2 - x) g_{2,3}(x, m_t^2) - (4m_t^2 - m_H^2) g_4(m_H^2, m_t^2) \\
C(x, y, z, m_t^2) &= (d-2) g_{5,6}(x, m_t^2) - (d-2) \frac{y+z}{x} g_{7,8}(x, m_H^2, m_t^2)
\end{aligned}
$$

Where $g_{i,j}$ can be either g_i or g_j depending on the argument x.

We define the normalized form factor as f_{212}. This is what we will be providing as input to pySecDec. Substituting the definitions for $A(), B()$ and $C()$ functions, we can express our normalized form factor f_{212} as linear combination of the 14 Integrals with 14 coefficients. We call these coefficients as P coefficients.

[5] Refer to the appendix: A, for the exact expressions of the T and N coefficients

The 14 P coefficients $P_1, P_2, \cdots, P_{14}$, are defined as follows :

$$\begin{aligned}
f_{212}(s_{12}, s_{13}, s_{23}) = \Big[\; & P_1 \cdot g_1(m_t^2) + P_2 \cdot g_2(s_{12}, m_t^2) \\
& + P_3 \cdot g_2(s_{13}, m_t^2) + P_4 \cdot g_3(s_{23}, m_t^2) + P_5 \cdot g_4(m_H^2, m_t^2) \\
& + P_6 \cdot g_5(s_{12}, m_t^2) + P_7 \cdot g_5(s_{13}, m_t^2) + P_8 \cdot g_6(s_{23}, m_t^2) \\
& + P_9 \cdot g_7(s_{12}, m_H^2, m_t^2) + P_{10} \cdot g_7(s_{13}, m_H^2, m_t^2) + P_{11} \cdot g_8(s_{23}, m_H^2, m_t^2) \\
& + P_{12} \cdot g_9(s_{12}, s_{23}, m_H^2, m_t^2) + P_{13} \cdot g_9(s_{12}, s_{13}, m_H^2, m_t^2) \\
& + P_{14} \cdot g_9(s_{23}, s_{13}, m_H^2, m_t^2) \;\Big]
\end{aligned}$$

The P Coefficients are obtained in terms of the T Coefficients as following :

$$P_1 = T_1$$

$$P_2 = T_2 \cdot (4m_t^2 - s_{12})$$

$$P_3 = T_3 \cdot (4m_t^2 - s_{13})$$

$$P_4 = T_4 \cdot (4m_t^2 - s_{23})$$

$$P_5 = (T_1 - T_2 - T_3 - T_4) \cdot (4m_t^2 - s_{12} - s_{13} - s_{23}) \quad \because (m_H^2 = s_{12} + s_{13} + s_{23})$$

$$P_6 = T_5 \cdot (d - 2)$$

$$P_7 = T_6 \cdot (d - 2)$$

$$P_8 = T_7 \cdot (d - 2)$$

$$P_9 = T_8 - T_5 \cdot (d - 2) \cdot \frac{(s_{13} + s_{23})}{s_{12}}$$

$$P_{10} = T_9 - T_6 \cdot (d - 2) \cdot \frac{(s_{23} + s_{12})}{s_{13}}$$

$$P_{11} = T_{10} - T_7 \cdot (d - 2) \cdot \frac{(s_{13} + s_{12})}{s_{23}}$$

$$P_{12} = T_{11}$$

$$P_{13} = T_{12}$$

$$P_{14} = T_{13}$$

The exact expressions for all the P and T coefficients can be viewed in the mathematica notebook for the F_{212} form factor at the Git-hub repository for this project : PSD_gggH1L

Now we have completely defined f_{212} form factor, ready for input to the pySecDec. We can deal with the f_{311} and f_{332} during the pySecDec implementation part. Let's focus on the F_{312} now. We again use the definition from [1]. Similarly as for F_{212}, we define temporary set of N Coefficients such that form factor F_{312} can be expressed as follows :

$$
\begin{aligned}
F_{312}(s_{12}, s_{13}, s_{23}) = f^{abc} C_\epsilon \cdot 2 m_t e g_{ht} g_s^3 \cdot \Big[& N_1 \cdot A(m_H^2, m_t^2) \\
& + N_2 \cdot B(s_{12}, m_H^2, m_t^2) \\
& + N_3 \cdot C(s_{12}, s_{13}, s_{23}, m_t^2) \\
& + N_4 \cdot g_7(s_{12}, m_H^2, m_t^2) \\
& + N_5 \cdot g_9(s_{12}, s_{23}, m_H^2, m_t^2) \\
& + N_{11} \cdot A(m_H^2, m_t^2) \\
& + N_{21} \cdot B(s_{13}, m_H^2, m_t^2) \\
& + N_{31} \cdot C(s_{13}, s_{23}, s_{12}, m_t^2) \\
& + N_{41} \cdot g_7(s_{13}, m_H^2, m_t^2) \\
& + N_{51} \cdot g_9(s_{13}, s_{12}, m_H^2, m_t^2) \\
& + N_{12} \cdot A(m_H^2, m_t^2) \\
& + N_{22} \cdot B(s_{23}, m_H^2, m_t^2) \\
& + N_{32} \cdot C(s_{23}, s_{12}, s_{13}, m_t^2) \\
& + N_{42} \cdot g_7(s_{23}, m_H^2, m_t^2) \\
& + N_{52} \cdot g_9(s_{23}, s_{13}, m_H^2, m_t^2) \Big] \\
\Rightarrow F_{312}(s_{12}, s_{13}, s_{23}) = f^{abc} C_\epsilon \cdot 2 m_t e g_{ht} g_s^3 \cdot f_{312}(s_{12}, s_{13}, s_{23})
\end{aligned}
$$

Where we have defined the normalized form factor as f_{312} which is what we will be providing as input to pySecDec. Using the definitions of $A(), B()$ and $C()$, we can express our normalized form factor f_{312} as linear combination of the 14 fntegrals with 14 coefficients. We call these 14 coefficients as M coefficients. The 14 M coefficients $M_1, M_2, \cdots, M_{14}$ are defined as follows :

$$
\begin{aligned}
f_{312}(s_{12}, s_{13}, s_{23}) = \Big[\; & M_1 \cdot g_1(m_t^2) + M_2 \cdot g_2(s_{12}, m_t^2) \\
& + M_3 \cdot g_2(s_{13}, m_t^2) + M_4 \cdot g_3(s_{23}, m_t^2) + M_5 \cdot g_4(m_H^2, m_t^2) \\
& + M_6 \cdot g_5(s_{12}, m_t^2) + M_7 \cdot g_5(s_{13}, m_t^2) + M_8 \cdot g_6(s_{23}, m_t^2) \\
& + M_9 \cdot g_7(s_{12}, m_H^2, m_t^2) + M_{10} \cdot g_7(s_{13}, m_H^2, m_t^2) + M_{11} \cdot g_8(s_{23}, m_H^2, m_t^2) \\
& + M_{12} \cdot g_9(s_{12}, s_{23}, m_H^2, m_t^2) + M_{13} \cdot g_9(s_{12}, s_{13}, m_H^2, m_t^2) \\
& + M_{14} \cdot g_9(s_{23}, s_{13}, m_H^2, m_t^2) \; \Big]
\end{aligned}
$$

The M coefficients in terms of N coefficients are expressed as follows :

$$M_1 = N_1 + N_{11} + N_{12}$$

$$
\begin{aligned}
M_2 &= (4m_t^2 - s_{12}) \cdot N_2 \\
M_3 &= (4m_t^2 - s_{13}) \cdot N_{21} \\
M_4 &= (4m_t^2 - s_{23}) \cdot N_{22} \\
M_5 &= (4m_t^2 - s_{12} - s_{13} - s_{23}) \cdot (N_1 + N_{11} + N_{12} - N_2 - N_{21} - N_{22}) \\
M_6 &= (d-2) \cdot N_3 \\
M_7 &= (d-2) \cdot N_{31}
\end{aligned}
$$

$$M_8 = (d-2) \cdot N_{32}$$

$$M_9 = N_4 - N_3 \cdot \frac{(d-2)(s_{13}+s_{23})}{s_{12}}$$

$$M_{10} = N_{41} - N_{31} \cdot \frac{(d-2)(s_{12}+s_{23})}{s_{13}}$$

$$M_{11} = N_{42} - N_{32} \cdot \frac{(d-2)(s_{12}+s_{13})}{s_{23}}$$

$$M_{12} = N_5$$

$$M_{13} = N_{51}$$

$$M_{14} = N_{52}$$

The exact expressions for all the M and N coefficients can be viewed in the mathematica notebook for the F_{312} form factor at the Git-hub repository for this project : PSD_gggH1L

We have completely defined both of the required form factors and are ready to provide them as input to pySecDec for numerical evaluation. But before that, we first set up the equation for matrix element squared with the normalized form factors and also investigate the analytic expressions in the Heavy Top quark Limit, which we will use for verification purpose.

4 Analytic Expressions

Once we have form factors in terms of the master integrals, all left to do is to substitute them into equation (II.48) to calculate the matrix element squared. One thing to note here is that, because we will be working with normalized form factors in pySecDec, We have to modify the equation (II.48) for normalized form factors.

The normalization factor for all form factors is $K = f^{abc} C_\epsilon \cdot 2 m_t e g_{ht} g_s^3$. Therefore, we will normalize the amplitude squared with $|K|^2$.

$$K = f^{abc} C_\epsilon \cdot 2 m_t e g_{ht} g_s^3 = f^{abc} \frac{1}{(4\pi)^2} \cdot \frac{2 m_t^2}{v} g_s^3$$

$$\Rightarrow \quad |K|^2 = N_c (N_c^2 - 1) \cdot \frac{m_t^4 \alpha_s^3}{\pi v^2}$$

where we have used the relations :

$$C_\epsilon = \frac{1}{(4\pi)^2} + \mathcal{O}(\epsilon), \quad e \cdot g_{ht} = \frac{m_t}{v}, \quad g_s^2 = 4\pi\alpha_s,$$

For the color algebra, we know

$$Tr(F^a F^b) = 2 \cdot T_R \cdot N_c \delta^{ab}$$

set $a = b$ and sum over, also use $T_R = 1/2$

$$\Rightarrow \quad |f^{abc}|^2 = N_c (N_c^2 - 1)$$

Therefore, we can now write Equation (II.48) with normalized form factors as follows :

$$\begin{aligned}\frac{|\mathcal{M}_{gg\to gH}|^2}{|K|^2} = \frac{1}{4} \cdot \Big[& \frac{(d-2)|f_{212}|^2 s_{12}^3 s_{23}}{s_{13}} + f_{212} f_{332}^* s_{12} s_{23}^2 + f_{212} f_{311}^* s_{12}^2 s_{13} + (d-2) f_{212} f_{312}^* s_{12}^2 s_{23} \\ & + f_{332} f_{212}^* s_{12} s_{23}^2 + \frac{(d-2)|f_{332}|^2 s_{13} s_{23}^3}{s_{12}} + f_{332} f_{311}^* s_{13}^2 s_{23} + (d-2) f_{332} f_{312}^* s_{13} s_{23}^2 \\ & + f_{311} f_{212}^* s_{12}^2 s_{13} + f_{311} f_{332}^* s_{13}^2 s_{23} + \frac{(d-2)|f_{311}|^2 s_{12} s_{13}^3}{s_{23}} + (d-2) f_{311} f_{312}^* s_{12} s_{13}^2 \\ & + (d-2) f_{312} f_{212}^* s_{12}^2 s_{23} + (d-2) f_{312} f_{332}^* s_{13} s_{23}^2 + (d-2) f_{312} f_{311}^* s_{12} s_{13}^2 \\ & + (3d-8)|f_{312}|^2 s_{12} s_{13} s_{23} \Big]\end{aligned} \tag{II.48a}$$

This is the general expression for amplitude squared. (axial gauge expression has been used for polarization sums. for more details : [1])

Once, we have numerically evaluated the form factors, we can substitute them in above expressions to get final results. To verify our numerical results, we will use the analytic expression for the amplitude squared, in the Heavy Top quark Limit. We will go over the details of Heavy Top quark Limit(HTL) during the implementation part. For now, we use the expression for amplitude squared in HTL, from [1].

$$|\mathcal{M}_{gg\to gH}|^2 = \frac{4}{9} \cdot \frac{N_c (N_c^2 - 1) \alpha_s^3}{\pi v^2} \cdot \frac{(m_H^8 + s_{12}^4 + s_{13}^4 + s_{23}^4)}{s_{12} s_{13} s_{23}} + \mathcal{O}(\epsilon)$$

We normalize it with $|K|^2$ to compare the results with (II.48a).

$$\frac{|\mathcal{M}_{gg\to gH}|^2}{|K|^2} = \frac{4}{9m_t^4} \cdot \frac{(m_H^8 + s_{12}^4 + s_{13}^4 + s_{23}^4)}{s_{12}s_{13}s_{23}} + \mathcal{O}(\epsilon)$$

This will be the final analytic Expression we will use for verification of our numerical results. We will call this the normalized amplitude squared. The individual form factors can also be verified in HTL. They will be expressed as follows :

$$f_{212} = \frac{4}{3m_t^2 s_{23}}, \quad f_{311} = \frac{4}{3m_t^2 s_{12}}, \quad f_{332} = \frac{4}{3m_t^2 s_{13}}, \quad f_{312} = \frac{4}{3m_t^2} \cdot \left(\frac{1}{s_{12}} + \frac{1}{s_{13}} + \frac{1}{s_{23}}\right)$$

Substituting the above form factors in (II.48a) lead us back to the normalized amplitude squared.

We can also evaluate the helicity amplitudes. Because each gluon can have two helicities, we have $2^3 = 8$ helicity amplitudes, although they are not all independent. They are all related by parity, such that only half of them are now independent. Furthermore, thanks to the permutation invariance of the amplitude, we finally have only two independent helicity amplitudes.

$$\mathcal{M}_{gg\to gH}^{h_1h_2h_3} = -\mathcal{M}_{gg\to gH}^{-h_1-h_2-h_3}$$

$$\mathcal{M}_{gg\to gH}^{++-}(s_{12}, s_{13}, s_{23}) = \mathcal{M}_{gg\to gH}^{+-+}(s_{13}, s_{23}, s_{12}) = \mathcal{M}_{gg\to gH}^{-++}(s_{23}, s_{12}, s_{13})$$

The two independent helicity amplitudes are represented in terms of the normalized form factors as follows : (Upto a phase factor)

$$\mathcal{M}_{gg\to gH}^{+++} = \frac{\sqrt{s_{12}s_{13}s_{23}}}{\sqrt{2}} \cdot \left(\frac{s_{12}}{2s_{13}} f_{212} + \frac{s_{23}}{2s_{12}} f_{332} + \frac{s_{13}}{2s_{23}} f_{311} + f_{312}\right)$$
$$\mathcal{M}_{gg\to gH}^{++-} = \sqrt{\frac{s_{12}s_{23}}{s_{13}}} \cdot \frac{s_{12}}{2\sqrt{2}} \cdot f_{212}$$

We can now express the normalized amplitude squared as sum of helicity amplitudes squared.

$$\frac{|\mathcal{M}_{gg\to gH}|^2}{|K|^2} = 2 \cdot \left(|\mathcal{M}_{gg\to gH}^{+++}|^2 + |\mathcal{M}_{gg\to gH}^{++-}|^2 + |\mathcal{M}_{gg\to gH}^{+-+}|^2 + |\mathcal{M}_{gg\to gH}^{-++}|^2\right) \tag{II.54a}$$

We have now established all the theoretical framework we will need to feed our example to pySecDec for numerical evaluation. We can head on over to the implementation part.

5 Implementation in pySecDec

All the files required for implementing our example in pySecDec can be found in the gggH1L folder at the Git-hub repository for this project : PSD_gggH1L [6]

We need to numerically evaluate the form factors with the *sum_package*. We have the form factors as linear combination of integrals with corresponding coefficients. Therefore, first we need to define all the 14 loop integrals in pySecDec. We can define them using any of the pySecDec functions, but using the propagators is convenient, because then we will just have to change the powers of propagators and pySecDec will take care of the rest. Extra care should be taken for defining the crossed integrals, depending on the legs interchanged. The integrals are ordered according to the previously established order only. IntegralsF212.py and IntegralsF312.py are both identical files with same list of integrals defined, which will be used for the form factors F_{212} and F_{312}.

CoefficientsF212.py and CoefficientsF312.py files have the corresponding coefficients defined in order of the integrals. The coefficients are calculated in the mathematica notebooks and copied using the raw input form. The numerator and denominator of coefficients are provided separately to the pySecDec. Key thing to remember here is that, coefficients should only be in terms of our real parameters $s_{12}, s_{13}, s_{23}, m_t^2, m_H^2$ and ϵ. Therefore, first substitute $d = 4 - 2\epsilon$ before providing them to pySecDec. Mind the order of the real parameters provided, same should be maintained for generation and integration. If we change it during the integration, we will get the different form factors, which we will get to later.

Once the integral coefficient files are ready, we can call the *sum_package* through the corresponding generate files for each form factor. It will provide the form factors as input to pySecDec, which we can then compile. After compilation, only job left to do is to integrate at a given set of conditions and match the results.

integrate_gggH1L.py file will perform the integration at given values of the real parameters and return the coefficient of ϵ^0 term, for all the form factors. Because our Amplitude is non-divergent, coefficients of ϵ^{-1} and ϵ^{-2} are just zero and we ignore them. Now, for the form factors f_{212} and f_{312}, we should maintain the correct order of real parameters, throughout all the files, to get correct results. However, because the form factors f_{311} and f_{332} are related to f_{212} through in-variance of amplitude under permutation, what we do is, we integrate the f_{212} two times again, at different order of real parameters, corresponding to the definitions f_{311} and f_{332}. This gives us the numerical results for f_{311}, f_{332} and thus, we have numerical results for all of the form factors. We use these values and substitute them in the equations (II.48a),(II.54a) with $d \to 4$ to get the numerical results for the normalized amplitude squared. We can also evaluate the analytic result for the same in HTL, and verify our numerical results.

Important thing while providing the values of real parameters for evaluation is that, they all must be in the physical scattering regions. The physical scattering region is given by :

$$s_{12} > 0, s_{13} < 0, s_{23} < 0 \;\&\; m_H^2 = s_{12} + s_{13} + s_{23}$$

We verify our results in the Heavy Top quark Limit(HTL), where $m_t \to \infty$ such that, $m_t^2 >> |s_{ij}|$ & $m_t^2 >> m_H^2$. In practice, to avoid large numerical uncertainties, we keep $m_t^2 = 1$ and choose the other real parameters to be very small ($<< 1$) such that the ratio of scales is still small. Refer to the appendix: B, for evaluation of integrals when large scale differences are present.

[6] https://github.com/CpSquared/PSD_gggH1L

Now we have all the files ready to execute, we can try to verify our results in the HTL. We will choose a set of conditions as follows and integrate to obtain results :

$$[s_{12}, s_{13}, s_{23}, m_t^2, m_H^2] = [0.0009, -0.0003, -0.000442873775, 1.00, 0.000157126225]$$

The explicit conditions we use for the integration are :

```
Integrator = Qmc  verbose=True  epsrel=1e-4  epsabs=1e-14
```

Integrating the form factors [7] we Obtain :

```
Numerical Results For All Normalized Form Factors upto a phase factor :
Normalized F212 :  3010.6672140051046 - 2.1398787465140891e-6*I
Normalized F311 :  -1481.49506223742628 + 9.2121555443932746 1e-7*I
Normalized F332 :  4444.48518094738301 - 3.11541162364117957e-6*I
Normalized F312 :  5973.69066508077594 + 0.0000168173049560099 87*I
Analytic results of all Normalized Form Factors with heavy top quark limit
Analytic Result Normalized F212 :  -3010.63961923086
Analytic Result Normalized F311 :  1481.4814814814813
Analytic Result Normalized F332 :  -4444.444444444444
Analytic Result Normalized F312 :  -5973.602582193824
```

Performing the amplitude calculation, we get following results :

Numerical Result of The Normalized Ampltiude Squared with eq (II.48a)
$|M_{-}(gg->gH)|^2$: 0.00261402286263782088

Numerical Result of The Normalized Ampltiude Squared with eq (II.54a)
$|M_{-}(gg->gH)|^2$: 0.00261402286263782093

Analytic Result of The Normalized Ampltiude Squared
$|M_{-}(gg->gH)|^2$: 0.002613976041276658

We can go ahead and perform some detailed error analysis on the numerical results obtained, to know more about the accuracy of our full numerical results with respect to the analytical counterparts in HTL.

[7] The `*I` represents multiplication with imaginary number i

6 Error analysis & Conclusion

The final amplitude expression we use for numerical evaluation is :

$$\mathcal{A} = 2 \cdot \left(|\mathcal{M}^{+++}_{gg\to gH}|^2 + |\mathcal{M}^{++-}_{gg\to gH}|^2 + |\mathcal{M}^{+-+}_{gg\to gH}|^2 + |\mathcal{M}^{-++}_{gg\to gH}|^2 \right) \tag{II.54a}$$

However, the helicity amplitudes themselves depend on form factors, which have numerical uncertainties associated with them, provided by pySecDec. We use these errors for form factors to calculate the net propagating numerical error in the final amplitude evaluation $\Delta\mathcal{A}$. Thus, we write out Numerical results for amplitude as :

$$\text{Numerical amplitude result} = \mathcal{A} \pm \Delta\mathcal{A}$$

We also calculate a few other measures :

$$\%\ \text{Numerical error} = \frac{\Delta\mathcal{A}}{\mathcal{A}} \times 100$$

and if $\mathcal{A}_0$ is the analytical amplitude(in HTL) value at corresponding kinematic point, we also calculate :

$$\%\ \text{Deviation} \ = \frac{\mathcal{A} - \mathcal{A}_0}{\mathcal{A}_0} \times 100$$

We will perform numerical evaluation at kinematic points scaled with m_t^2 as follows :

$$[s_{12}, s_{13}, s_{23}, m_t^2, m_H^2] = [0.0009 \cdot m_t^2, -0.0003 \cdot m_t^2, -0.000442873775 \cdot m_t^2, m_t^2, 0.000157126225 \cdot m_t^2]$$

Consequently, the ratios $\frac{|s_{ij}|}{m_t^2} \sim 10^{-4}$ and $\frac{m_H^2}{m_t^2} \sim 10^{-4}$ stay constant, and stay in the heavy top limit, ir-respective of the value of m_t^2. We will go ahead and vary the value of m_t^2 to observe the effect in numerical error propagated by pySecDec for final amplitude. Let's review the behaviour of error with m_t^2 scaling in Table 1.

Specifically, the numerical results at $m_t^2 = 1$.

`Numerical Result of The Normalized Ampltiude Squared with eq (II.54a)`
$|M\_(gg->gH)|^2$: `(2.61402286630 +/- 0.00000000264) *` 10^{-3}

`Analytic Result of The Normalized Ampltiude Squared in Heavy Top Limit`
$|M\_(gg->gH)|^2$: `(2.61397604127) *` 10^{-3}

To conclude, if the m_t^2 is larger than other invariants by a factor $\sim 10^4$, then DO NOT set the value of m_t^2 greater than 10^8, it will lead to large numerical errors. Preferred value is anything less than or $\sim 10^4$ for lowest numerical errors. Then, the relative difference between the full numerical results and analytical results in HTL is of order $\sim 10^{-4}$.

The arguements `epsrel` and `epsabs` control the relative and absolute inaccuracies of the numerical result, and our numerical results are observed to obey these in satisfactory bounds. To extend on to a complicated multi-loop example with the help of these weighted sums, one should take care of the error bounds accordingly.

m_t^2	$\mathcal{A}$	$\Delta\mathcal{A}$	$\mathcal{A}_0$	% Numerical error	% Deivation
10^{-12}	$2.61402286486 \cdot 10^{9}$	$1.10325112235 \cdot 10^{0}$	$2.61397604127 \cdot 10^{9}$	$4.2205106053 \cdot 10^{-8}$	$1.79127841460 \cdot 10^{-3}$
10^{-10}	$2.61402286507 \cdot 10^{7}$	$1.12967622883 \cdot 10^{-2}$	$2.61397604127 \cdot 10^{7}$	$4.3216004111 \cdot 10^{-8}$	$1.79128625618 \cdot 10^{-3}$
10^{-8}	$2.61402286484 \cdot 10^{5}$	$1.14097376996 \cdot 10^{-4}$	$2.61397604127 \cdot 10^{5}$	$4.3648193951 \cdot 10^{-8}$	$1.79127778127 \cdot 10^{-3}$
10^{-6}	$2.61402286815 \cdot 10^{3}$	$1.00811514475 \cdot 10^{-6}$	$2.61397604127 \cdot 10^{3}$	$3.8565658971 \cdot 10^{-8}$	$1.79140435844 \cdot 10^{-3}$
10^{-4}	$2.61402286225 \cdot 10^{1}$	$1.17774107579 \cdot 10^{-8}$	$2.61397604127 \cdot 10^{1}$	$4.505473509 \cdot 10^{-8}$	$1.79117853416 \cdot 10^{-3}$
10^{-2}	$2.61402286406 \cdot 10^{-1}$	$1.02678005700 \cdot 10^{-10}$	$2.61397604127 \cdot 10^{-1}$	$3.9279689215 \cdot 10^{-8}$	$1.79124787455 \cdot 10^{-3}$
1	$2.61402286630 \cdot 10^{-3}$	$2.64465133233 \cdot 10^{-12}$	$2.61397604127 \cdot 10^{-3}$	$1.0117169847 \cdot 10^{-7}$	$1.79133366574 \cdot 10^{-3}$
10^{2}	$2.61402287068 \cdot 10^{-5}$	$2.75310779611 \cdot 10^{-14}$	$2.61397604127 \cdot 10^{-5}$	$1.0532072336 \cdot 10^{-7}$	$1.79150118589 \cdot 10^{-3}$
10^{4}	$2.61402285788 \cdot 10^{-7}$	$2.82184625032 \cdot 10^{-16}$	$2.61397604127 \cdot 10^{-7}$	$1.0795032805 \cdot 10^{-7}$	$1.79101118864 \cdot 10^{-3}$
10^{6}	$2.61400648697 \cdot 10^{-9}$	$2.12661964630 \cdot 10^{-14}$	$2.61397604127 \cdot 10^{-9}$	$8.13547960536 \cdot 10^{-4}$	$1.16472764313 \cdot 10^{-3}$
10^{8}	$2.65071422899 \cdot 10^{-11}$	$6.37139264259 \cdot 10^{-13}$	$2.61397604127 \cdot 10^{-11}$	$2.40365127742 \cdot 10^{0}$	$1.40545235067 \cdot 10^{0}$
10^{10}	$2.03581279319 \cdot 10^{-9}$	$2.94981372513 \cdot 10^{-9}$	$2.61397604127 \cdot 10^{-13}$	$1.44896118886 \cdot 10^{2}$	$7.78718459332 \cdot 10^{5}$

Table 1: Error analysis with m_t^2 scaling

This completes the successful demonstration of *sum_package* with pySecDec to numerically evaluate the 1-Loop amplitude for $gg \to gH$, by using the linear combination of master integrals. This will build the ground for the multi-loop calculations of crucial processes which will play an important role in the collider phenomenology after the high luminosity upgrade at the LHC .

7 Acknowledgement

I would like to formally express my gratitude towards Prof. Dr. Gudrun Heinrich for her Invaluable guidance during this project. During this project, I had the incredible opportunity to attend the Collider Phenomenology Lectures, which were of great help with this project and educated me on a wide array of topics for precision calculations. I would also like to thank the German Academic Exchange Service (DAAD) for offering the DAAD WISE Scholarship and supporting undergraduates at the research frontier.

References

[1] M. Boggia, J. M. Cruz-Martinez, H. Frellesvig, E. W. N. Glover, R. Gomez-Ambrosio, G. Gonella, Y. Haddad, A. Ilnicka, S. P. Jones and Z. Kassabov, *et al.* J. Phys. G **45**, no.6, 065004 (2018) doi:10.1088/1361-6471/aab812 [arXiv:1711.09875 [hep-ph]].

[2] S. Borowka, G. Heinrich, S. Jahn, S. P. Jones, M. Kerner, J. Schlenk and T. Zirke, Comput. Phys. Commun. **222**, 313-326 (2018) doi:10.1016/j.cpc.2017.09.015 [arXiv:1703.09692 [hep-ph]].

[3] S. Borowka, G. Heinrich, S. Jahn, S. P. Jones, M. Kerner and J. Schlenk, Comput. Phys. Commun. **240**, 120-137 (2019) doi:10.1016/j.cpc.2019.02.015 [arXiv:1811.11720 [physics.comp-ph]].

[4] G. Heinrich, S. Jahn, S. P. Jones, M. Kerner, F. Langer, V. Magerya, A. Põldaru, J. Schlenk and E. Villa, [arXiv:2108.10807 [hep-ph]].

[5] A. G. Grozin, Int. J. Mod. Phys. A **26**, 2807-2854 (2011) doi:10.1142/S0217751X11053687 [arXiv:1104.3993 [hep-ph]].

[6] G. Heinrich, Phys. Rept. **922**, 1-69 (2021) doi:10.1016/j.physrep.2021.03.006 [arXiv:2009.00516 [hep-ph]].

[7] M. Beneke and V. A. Smirnov, Nucl. Phys. B **522**, 321-344 (1998) doi:10.1016/S0550-3213(98)00138-2 [arXiv:hep-ph/9711391 [hep-ph]].

8 Appendix : A

- T Coefficient expressions :

$$
\begin{aligned}
T_1 &= \frac{8(d-4)(s_{12}^2 - s_{13}s_{23})}{s_{12}s_{23}(d-3)(d-2)(s_{12}+s_{23})(s_{12}+s_{13})} \\
T_2 &= \frac{-4(d-4)}{s_{12}s_{23}(d-3)(d-2)} \\
T_3 &= \frac{-4s_{13}\big((d-4)s_{12}^2 + 2ds_{12}s_{23} + ds_{23}^2\big)}{s_{12}^2 s_{23}(d-3)(d-2)(s_{12}+s_{23})^2} \\
T_4 &= \frac{-4\big((d-4)s_{12}^2 + 2ds_{12}s_{13} + ds_{13}^2\big)}{s_{12}^2(d-3)(d-2)(s_{12}+s_{13})^2} \\
T_5 &= \frac{-(d-4)(s_{13}^2 + s_{23}^2)}{s_{13}s_{23}^2(d-3)(d-2)} \\
T_6 &= \frac{-s_{13}^2\big((d-4)s_{12}^2 + ds_{23}^2\big)}{s_{12}^3 s_{23}^2(d-3)(d-2)} \\
T_7 &= \frac{-s_{23}\big((d-4)s_{12}^2 + ds_{13}^2\big)}{s_{12}^3 s_{13}(d-3)(d-2)} \\
T_8 &= \frac{2(s_{13}+s_{23})\big((d-2)s_{12} - 8m_t^2\big)}{s_{12}^2 s_{23}(d-2)} \\
T_9 &= \frac{2\big((d-2)s_{12}(s_{12}^2 - s_{23}^2) - 8m_t^2(s_{12}^2 - 2s_{12}s_{23} - s_{23}^2)\big)}{s_{12}^2 s_{23}(s_{12}+s_{23})(d-2)} \\
T_{10} &= \frac{2\big((d-2)s_{12}(s_{12}^2 - s_{13}^2) - 8m_t^2(s_{12}^2 - 2s_{12}s_{13} - s_{13}^2)\big)}{s_{12}^2 s_{23}(s_{12}+s_{13})(d-2)} \\
T_{11} &= \frac{(d-4)s_{12}s_{23} + (d-3)s_{12}s_{13} - 4m_t^2 s_{13}}{s_{12}s_{13}(d-3)} \\
T_{12} &= \frac{s_{13}\big((d-4)s_{12}s_{13} + (d-3)s_{12}s_{23} - 4m_t^2 s_{23}\big)}{s_{12}s_{23}^2(d-3)} \\
T_{13} &= \frac{s_{13}\big(-(d-3)s_{12}^2 + ds_{13}s_{23} + 12m_t^2 s_{12}\big)}{s_{12}^3(d-3)}
\end{aligned}
$$

- N Coefficient expressions :

$$
\begin{aligned}
N_1 &= \frac{8(d-4)}{(d-3)(d-2)(s_{12}s_{13} + s_{13}s_{23})} \\
N_2 &= \frac{4\big((d-2)s_{13}^2 + 2ds_{13}s_{23} + (d-2)s_{23}^2\big)}{s_{13}s_{23}(d-3)(d-2)(s_{13}+s_{23})^2} \\
N_3 &= \frac{s_{12}(s_{13}^2 + s_{23}^2)}{s_{13}^2 s_{23}^2(d-3)} \\
N_4 &= \frac{4\big((d-2)(s_{13}+s_{23}) - 8m_t^2\big)}{s_{12}(d-2)(s_{13}+s_{23})} \\
N_5 &= \frac{\big((d-3)s_{13}^2 - (d-2)s_{12}s_{23} - 4m_t^2 s_{13}\big)}{s_{13}^2(d-3)}
\end{aligned}
$$

$$N_{i1} = N_i(s_{12} \to s_{13}, s_{13} \to s_{23}, s_{23} \to s_{12}) \qquad N_{i2} = N_i(s_{12} \to s_{23}, s_{13} \to s_{12}, s_{23} \to s_{13})$$

where $i = \{1, 2, 3, 4, 5\}$

9 Appendix : B

• Evaluation of master integrals with expansion by regions :

Expansion by regions is a systematic method to express the feynman integral as a series expansion in terms of a 'smallness parameter', which arises if there are large scale differences present between kinematic invariants or the masses of the integral. While evaluating the integrals regularly, pySecDec cannot handle large scale differences, which is why this particular method is very useful in such cases. Therefore, to evaluate the integrals in heavy top limit, we will employ this method and try to verify our results with the analytic expressions from [1]. For a brief review of this method, refer to the section 3.2.7 of [6] or the original article [7].

Because we are working in the HTL, the integrals are expanded in inverse powers of m_t^2. The analytic expressions for the master integrals [8] can be calculated to be as follows :

$$g_2(s_{12}, m_t^2) = \frac{-(6m_t^2 + s_{12})}{12m_t^4} + \epsilon \cdot \Big(\frac{\gamma_E(6m_t^2 + s_{12})}{12m_t^4} + \frac{-\big(s_{12} + (6m_t^2 + s_{12}) \cdot \ln \frac{1}{m_t^2}\big)}{12m_t^4}\Big)$$

$$g_5(s_{12}, m_t^2) = \frac{-(12m_t^2 + s_{12})}{24m_t^4} + \epsilon \cdot \Big(\frac{\gamma_E(12m_t^2 + s_{12})}{24m_t^4} + \frac{-\big(s_{12} + (12m_t^2 + s_{12}) \cdot \ln \frac{1}{m_t^2}\big)}{24m_t^4}\Big)$$

$$g_7(s_{12}, m_H^2, m_t^2) = \frac{-(m_H^2 + 12m_t^2 + s_{12})}{24m_t^4} + \epsilon \cdot \Big(\frac{\gamma_E(m_H^2 + 12m_t^2 + s_{12})}{24m_t^4} + \frac{-\big(m_H^2 + s_{12} + (m_H^2 + 12m_t^2 + s_{12}) \cdot \ln \frac{1}{m_t^2}\big)}{24m_t^4}\Big)$$

$$g_9(s_{12}, s_{23}, m_H^2, m_t^2) = \frac{1}{6m_t^4} + \epsilon \cdot \Big(\frac{1 + \ln \frac{1}{m_t^2} - \gamma_E}{6m_t^4}\Big)$$

where we have dropped the terms $\sim \epsilon \cdot \mathcal{O}(\frac{1}{m_t^6}), \quad \sim \mathcal{O}(\frac{1}{m_t^6})$ & $\sim \mathcal{O}(\epsilon^2)$

All the files required for the evaluation of these integrals using expansion by regions with pySecDec can be found in the `'Expansion by regions'` folder at the Git-hub repository for this Project : PSD_gggH1L

We can define the loop integrals in the same way as we did earlier, but in order to introduce the 'smallness parameter'-z, we multiply all the in-variants and masses (except m_t) with z. We will also add z to the list of real parameters, and provide our loop integral to the `loop_regions` method, which will expand the integral in different regions. We can then proceed to sum over all regions using the `sum_package`.

However, the $g_5(s_{12}, m_t^2)$ integral must be treated carefully, as it needs an additional regulator to evaluate it. The same will be notified by pySecDec, if the integral needs an additional regulator. We introduce the extra regulator n_1 in the powers of the propagators, by consequently dividing it with the prime numbers. Therefore, the modified conditions for $g_5(s_{12}, m_t^2)$ should look like following :
(remaining conditions same as other integrals)

```
 powerlist = ["1+n1", "1+ n1/2", "1+n1/3", "0+n1/5"]
regulators=["n1","eps"]  requested_orders = [0,1]
```

[8] Because the $g_1(m_t^2)$ integral has only a single scale dependence, we cannot expand it in small ratio of scales, thus we do not consider it.

Now we have all the integrals ready to evaluate, we can try to verify our results with the analytic expression mentioned above, in the HTL. We will choose a set of conditions as follows and integrate to obtain results :

$$[s_{12}, s_{13}, s_{23}, m_t^2, m_H^2] = [0.0009, -0.0003, -0.000442873775, 1.00, 0.000157126225]$$

The explicit conditions we use for the Integration are : (modify for $g_5(s_{12}, m_t^2)$ as mentioned earlier)

```
  expansion_by_regions_order = 1  requested_orders = [1]  z = 1
Integrator = Qmc  transform="korobov3"  fitfunction="polysingular"
```

Integrating $g_2(s_{12}, m_t^2)$ we Obtain :

```
Analytic Result :
eps^0 :  -0.5000749999999999
eps^1 :  0.288576123625634
Numerical Result :
eps^0 :  -0.500075000000000+0.000000000000000*I
          +/- 3.389926606603533e-17+1.358594763071333e-17*I
eps^1 :  0.288576123625634-0.000000000000000*I
          +/- 1.956787297766954e-17+7.843700577440028e-18*I
```

Similalrly, we can calculate for the other integrals and verify our procedure. The output summary for all the integrals is given in the `Output_Summary_exp_by_regions.txt` file in the `'Expansion by regions'` folder mentioned earlier.

We have successfully demonstrated the evaluation of master integrals using the method of expansion by regions in the heavy top limit with `pySecDec`.